

## Description of FEM theory for programming

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# 1 Interpolation Function and Gauss-Legendre Quadrature

## 1.1 Introduction

In the finite element method, it is necessary to express the physical quantities of any point in the element with nodal physical quantities using interpolation function  $[N]$ . For example, in the thermal conductivity analysis, a temperature at any point  $T$  can be expressed as follow.

$$T = N_1(a, b) \cdot \phi_i + N_2(a, b) \cdot \phi_j + N_3(a, b) \cdot \phi_k + N_4(a, b) \cdot \phi_l$$

where,  $\phi_i, \phi_j, \phi_k, \phi_l$  mean the temperature at node  $i, j, k, l$  for each. If vector expression is applied,

$$T = [N_1 \quad N_2 \quad N_3 \quad N_4] \{\phi\} = [N] \{\phi\}$$

where,  $\{\phi\}$  means nodal temperature vector. In this section, it is considered to use the Gauss-Legendre quadrature for 4 nodes element and to express the physical quantities using the normalized parameter coordinate  $(a, b)$  with range of  $[-1, 1]$ .

## 1.2 Interpolation function and its derivative

In actual calculation, since a mathematical expression and derivative of interpolation function are required, they will be derived. After this, the interpolation function  $N(a, b)$  is used, where  $(a, b)$  is normalized parameter coordinate with range of  $a = [-1, 1]$  and  $b = [-1, 1]$ .

### 1.2.1 Mathematical expression of interpolation function

For a 4 nodes element, following interpolation function can be applied.

$$\begin{aligned} N_1 &= \frac{1}{4}(1-a)(1-b) & N_2 &= \frac{1}{4}(1+a)(1-b) \\ N_3 &= \frac{1}{4}(1+a)(1+b) & N_4 &= \frac{1}{4}(1-a)(1+b) \end{aligned}$$

### 1.2.2 Derivative of interpolation function

The partial derivatives of  $N$  about  $a$  and  $b$  can be obtained as follows.

$$\begin{aligned} \frac{\partial N_1}{\partial a} &= -\frac{1}{4}(1-b) & \frac{\partial N_2}{\partial a} &= +\frac{1}{4}(1-b) & \frac{\partial N_3}{\partial a} &= +\frac{1}{4}(1+b) & \frac{\partial N_4}{\partial a} &= -\frac{1}{4}(1+b) \\ \frac{\partial N_1}{\partial b} &= -\frac{1}{4}(1-a) & \frac{\partial N_2}{\partial b} &= -\frac{1}{4}(1-a) & \frac{\partial N_3}{\partial b} &= +\frac{1}{4}(1+a) & \frac{\partial N_4}{\partial b} &= +\frac{1}{4}(1-a) \end{aligned}$$

Next, above equations can be expressed using Jacobi matrix  $[J]$  as follows.

$$\begin{aligned} \begin{Bmatrix} \frac{\partial N_i}{\partial a} \\ \frac{\partial N_i}{\partial b} \end{Bmatrix} &= [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} & \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} &= [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial a} \\ \frac{\partial N_i}{\partial b} \end{Bmatrix} \\ [J] &= \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^4 \left( \frac{\partial N_i}{\partial a} x_i \right) & \sum_{i=1}^4 \left( \frac{\partial N_i}{\partial a} y_i \right) \\ \sum_{i=1}^4 \left( \frac{\partial N_i}{\partial b} x_i \right) & \sum_{i=1}^4 \left( \frac{\partial N_i}{\partial b} y_i \right) \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \\ [J]^{-1} &= \frac{1}{\det(J)} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} & \det(J) &= J_{11} \cdot J_{22} - J_{12} \cdot J_{21} \end{aligned}$$

From above,

$$\frac{\partial N_i}{\partial x} = \frac{1}{\det(J)} \left\{ J_{22} \frac{\partial N_i}{\partial a} - J_{12} \frac{\partial N_i}{\partial b} \right\} \quad \frac{\partial N_i}{\partial y} = \frac{1}{\det(J)} \left\{ -J_{21} \frac{\partial N_i}{\partial a} + J_{11} \frac{\partial N_i}{\partial b} \right\}$$

the each element of  $[J]$  can be shown as follows, where  $x_{i,j,k,l}$  and  $y_{i,j,k,l}$  means nodal coordinates which form a finite element.

$$\begin{aligned}
 J_{11} &= \frac{\partial N_1}{\partial a} x_i + \frac{\partial N_2}{\partial a} x_j + \frac{\partial N_3}{\partial a} x_k + \frac{\partial N_4}{\partial a} x_l \\
 J_{12} &= \frac{\partial N_1}{\partial a} y_i + \frac{\partial N_2}{\partial a} y_j + \frac{\partial N_3}{\partial a} y_k + \frac{\partial N_4}{\partial a} y_l \\
 J_{21} &= \frac{\partial N_1}{\partial b} x_i + \frac{\partial N_2}{\partial b} x_j + \frac{\partial N_3}{\partial b} x_k + \frac{\partial N_4}{\partial b} x_l \\
 J_{22} &= \frac{\partial N_1}{\partial b} y_i + \frac{\partial N_2}{\partial b} y_j + \frac{\partial N_3}{\partial b} y_k + \frac{\partial N_4}{\partial b} y_l
 \end{aligned}$$

From above, the elements of  $[J]$  were defined and derivative of  $[N]$  can be calculated. At this point, it is necessary to note that  $[J]$ ,  $[N]$  and its derivative are function of variables  $a$  and  $b$ .

### 1.3 Gauss-Legendre quadrature

Gauss-Legendre quadrature is used for integration.

#### 1.3.1 Surface integral

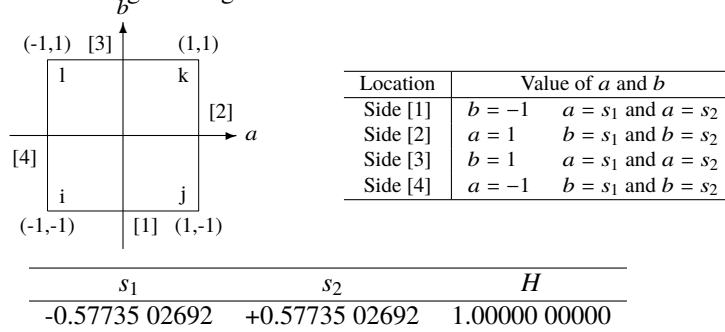
When 4 Gauss points ( $n = 2$ ) are considered in a finite element, the values of  $a$ ,  $b$  and weight  $H$  can be shown in below table. In this case, the approximation of integral value can be calculated as a summation of 4 times calculation depending on the coordinate  $(a, b)$ . Furthermore, since the weight  $H$  is equal to 1 for all coordinate, the calculation can be more simplified.

$$\int_{-1}^1 \int_{-1}^1 f(a, b) \cdot da \cdot db \doteq \sum_{i=1}^n \sum_{j=1}^n H_i \cdot H_j \cdot f(a_i, b_j) = \sum_{kk=1}^4 f(a_{kk}, b_{kk})$$

$i$	$j$	$a$	$b$	$H$	$kk$
1	1	-0.57735 02692	-0.57735 02692	1.00000 00000	1
1	2	+0.57735 02692	-0.57735 02692	1.00000 00000	2
2	1	+0.57735 02692	+0.57735 02692	1.00000 00000	3
2	2	-0.57735 02692	+0.57735 02692	1.00000 00000	4

#### 1.3.2 Curvilinear integral

In the thermal conductivity analysis, curvilinear integral is required on the element side which has heat transfer boundary condition. The method of curvilinear integral along the element side is shown below.



## 2 General equilibrium equation for static structural analysis using principle of virtual work

In this section, a derivation of general equilibrium equation for static structural analysis using principle of virtual work is described. Generally, in case that an elastic body subjected to an external force is under the equilibrium state, the virtual work due to the internal stresses in the body is equal to the virtual work due to the surface force or body force. This relationship can be expressed as follows.

$$\int_V \{\delta \epsilon\} \{\sigma\} dV = \int_A \{\delta u\} \{S\} dA + \int_V \{\delta u\} \{F\} dV$$

$\{\delta\epsilon\}$	: virtual strain in stress direction	$\{\sigma\}$	: internal stress in the body
$\{\delta u\}$	: virtual displacement in force direction	$\{S\}$	: surface force
$V$	: volume of a body	$\{F\}$	: body force (inertia force)
$A$	: area of surface force acts		

At this point, it is assumed that a body deems an finite element and strain  $\{\epsilon\}$  and displacement  $\{u\}$  at any point in a finite element can be expressed using nodal displacement  $\{u_{nd}\}$  as follows.

$$\begin{aligned}\{\epsilon\} &= [B]\{u_{nd}\} \\ \{u\} &= [N]\{u_{nd}\}\end{aligned}$$

where,  $[B]$  is a strain-displacement relationship matrix,  $[N]$  is an interpolation function matrix. And a stress-strain relationship of elastic body is applied shown below.

$$\{\sigma\} = [D_e]\{\epsilon - \epsilon_0\}$$

where,  $[D_e]$  is stress-strain relationship matrix,  $\{\epsilon_0\}$  is initial strain due to temperature change and so on. Next, it is assumed that  $\{F\}$  can be expressed using nodal inertia force  $\{w_{nd}\}$  as follow.

$$\{F\} = [N]\{w_{nd}\}$$

From above, left side of virtual work equation which means virtual work due to internal force becomes as follow.

$$\begin{aligned}\int_V \{\delta\epsilon\}\{\sigma\}dV &= \{\delta u_{nd}\}^T \int_V [B]^T \{\sigma\}dV \\ &= \{\delta u_{nd}\}^T \left( \int_V [B]^T [D_e] [B]dV \right) \{u_{nd}\} - \{\delta u_{nd}\}^T \left( \int_V [B]^T [D_e] \{\epsilon_0\}dV \right) \\ &= \{\delta u_{nd}\}^T [k] \{u_{nd}\} - \{\delta u_{nd}\}^T \{f_t\}\end{aligned}$$

And right side of virtual work equation which means virtual work due to external force becomes as follow.

$$\begin{aligned}\int_A \{\delta u\}\{S\}dA &= \{\delta u_{nd}\}^T \left( \int_A [N]^T \{S\}dA \right) = \{\delta u_{nd}\}^T \{f\} \\ \int_V \{\delta u\}\{F\}dV &= \{\delta u_{nd}\}^T \left( \int_V [N]^T [N]dV \right) \{w_{nd}\} = \{\delta u_{nd}\}^T \{f_b\}\end{aligned}$$

As a result, general form of stiffness equation can be obtained shown below.

$$[k]\{u_{nd}\} = \{f\} + \{f_t\} + \{f_b\}$$

$$\begin{aligned}[k] &= \int_V [B]^T [D_e] [B]dV && \text{(element stiffness matrix)} \\ \{f\} &= \int_A [N]^T \{S\}dA && \text{(nodal external force vector)} \\ \{f_t\} &= \int_V [B]^T [D_e] \{\epsilon_0\}dV && \text{(nodal force vector due to initial strain)} \\ \{f_b\} &= \left( \int_V [N]^T [N]dV \right) \{w_{nd}\} && \text{(nodal inertia force vector)}\end{aligned}$$

### 3 2D Frame Analysis

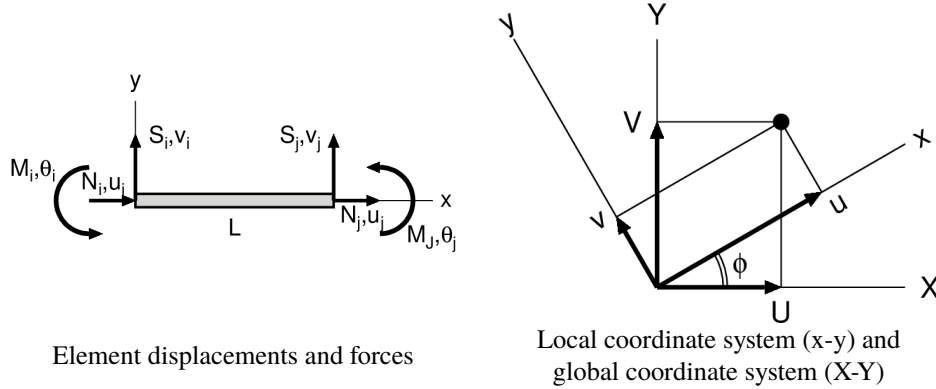
#### 3.1 Element stiffness equation

Element stiffness equation is shown below.

$$\{f\} = [k]\{u\}$$

$$\begin{Bmatrix} N_i \\ S_i \\ M_i \\ N_j \\ S_j \\ M_j \end{Bmatrix} = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix}$$

$EA$  axial rigidity       $N$  axial force       $u$  axial displacement (in x-direction)  
 $EI$  bending rigidity       $S$  Shearing force       $v$  deflection (in y-direction)  
 $L$  element length       $M$  bending moment       $\theta$  rotation



### 3.2 Coordinate transformation matrix

Coordinate transformation matrix from global coordinate system to local coordinate system is shown below.

$$\{u\} = [T]\{U\}$$

$$\begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 & 0 & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \\ \Theta_i \\ U_j \\ V_j \\ \Theta_j \end{Bmatrix}$$

### 3.3 Stiffness equation in global coordinate system

Stiffness equation in global coordinate system is shown below. This shall be assembled for all elements.

$$[K]\{U\} = \{F\} + \{F_t\} + \{F_b\}$$

$[K] = [T]^T [k] [T]$  : stiffness matrix in global coordinate system  
 $\{U\}$  : nodal displacement vector in global coordinate system  
 $\{F\}$  : nodal external force vector in global coordinate system  
 $\{F_t\} = [T]^T \{f_t\}$  : nodal thermal load vector in global coordinate system  
 $\{F_b\} = \{f_b\}$  : nodal inertia force vector in global coordinate system

$$\{f_t\} = \begin{Bmatrix} -EA \cdot \alpha \cdot \Delta T \\ 0 \\ 0 \\ EA \cdot \alpha \cdot \Delta T \\ 0 \\ 0 \end{Bmatrix} \quad \{f_b\} = \frac{\gamma A \ell}{2} \begin{Bmatrix} k_h \\ k_v \\ 0 \\ k_h \\ k_v \\ 0 \end{Bmatrix}$$

$EA$  : axial rigidity       $\alpha$  : thermal expansion coefficient       $\Delta T$  : temperature change  
 $\gamma$  : unit weight       $A$  : element section area       $\ell$  : element length  
 $k_h$  : horizontal acceleration       $k_v$  : vertical acceleration

Since element nodal thermal load vector  $\{f_t\}$  is a correction term of axial force of member, the coordinate transformation is required when assembled. Whereas, since element nodal inertia force vector  $\{f_b\}$  is defined using accelerations in global coordinate system, the coordinate transformation is not required when assembled. Furthermore, when section force is calculated from the solution of total stiffness equation, axial force shall be corrected as follows.

$$N'_i = N_i + EA \cdot \alpha \cdot \Delta T$$

$$N'_j = N_j - EA \cdot \alpha \cdot \Delta T$$

## 4 2D Truss Analysis

### 4.1 Element stiffness equation

The element stiffness matrix of 2D truss consists of the matrix which doesn't include the item of rotation or moment of 2D frame element stiffness matrix. The coordinate transformation matrix also doesn't include the item of rotation of that of 2D frame.

The element stiffness equation of 2D truss is shown below.

$$\begin{Bmatrix} N_i \\ S_i \\ N_j \\ S_j \end{Bmatrix} = \begin{bmatrix} EA/L & 0 & -EA/L & 0 \\ 0 & 0 & 0 & 0 \\ -EA/L & 0 & EA/L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

$EA$  axial rigidity       $N$  axial force       $u$  displacement in x-direction  
 $L$  element length       $S$  shearing force       $v$  displacement in y-direction

The coordinate transformation matrix of 2D truss is shown below.

$$\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{Bmatrix}$$

## 5 Grid Girder Analysis

### 5.1 Element stiffness equation

The relationships of section forces between grid girder structure and 2D frame structure are shown below.

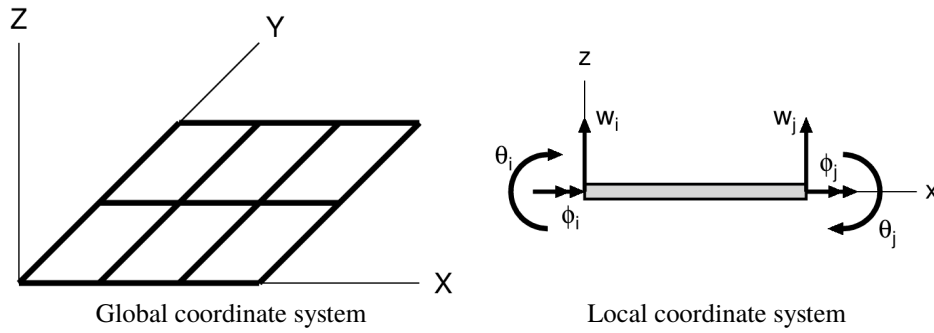
Grid girder	2D frame
Torsional moment	Axial force
Bending moment	Shearing force
Shearing force	Bending moment

The element stiffness equation of grid girder structure is shown below.

$$\begin{Bmatrix} T_i \\ M_i \\ Q_i \\ T_j \\ M_j \\ Q_j \end{Bmatrix} = \begin{bmatrix} GJ/L & 0 & 0 & -GJ/L & 0 & 0 \\ 0 & 4EI/L & -6EI/L^2 & 0 & 2EI/L & 6EI/L^2 \\ 0 & -6EI/L^2 & 12EI/L^3 & 0 & -6EI/L^2 & -12EI/L^3 \\ -GJ/L & 0 & 0 & GJ/L & 0 & 0 \\ 0 & 2EI/L & -6EI/L^2 & 0 & 4EI/L & 6EI/L^2 \\ 0 & 6EI/L^2 & -12EI/L^3 & 0 & 6EI/L^2 & 12EI/L^3 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \theta_i \\ w_i \\ \phi_j \\ \theta_j \\ w_j \end{Bmatrix}$$

$GJ$  torsional rigidity       $T$  torsional moment       $\phi$  rotation around x-axis  
 $EI$  bending rigidity       $M$  bending moment       $\theta$  rotation around y-axis  
 $L$  element length       $Q$  shearing force       $w$  deflection in z-direction

Since the coordinate transformation is carried out on X-Y plane, the coordinate transformation matrix is the same as that for 2D frame analysis.



(Reference) Torsional constant of rectangular cross section  $J$

The torsional constant  $J$  such as for concrete rectangular cross section can be calculated as follow.

$$J = \frac{hb^3}{3} \left\{ 1 - \frac{192b}{\pi^5 h} \sum_{n=1}^{\infty} \left[ \frac{1}{(2n-1)^5} \tanh \frac{(2n-1)\pi h}{2b} \right] \right\} \quad h \geq b$$

When the coefficient  $\eta$  is introduced, the relationship between coefficient  $\eta$  and the member dimension  $h/b$  becomes shown below.

$$J = \frac{hb^3}{\eta}$$

$h/b$	1	2	3	5	10	20
$\eta$	7.114	4.373	3.798	3.433	3.202	3.098

Above calculation was done by Python program shown below.

```
import numpy as np

### Result ###
# h/b  1.0    2.0    3.0    5.0    10.0   20.0
# eta  7.114  4.373  3.798  3.433  3.202  3.098
# J=h*b**3/eta

n=100
b=1.0
#hh=np.arange(1.0,21.0,1.0)
hh=np.array([1.0,2.0,3.0,5.0,10.0,20.0])
for idx,elem in enumerate(hh):
    h=elem
    s=0.0
    for i in range(1,n+1):
        s=s+1/(2*i-1)**5*np.tanh((2*i-1)*np.pi*h/2/b)
    x=1/3*(1-192/np.pi**5*b/h*s)
    print('{0:10.3f} {1:10.3f}'.format(h/b,1/x))
```

## 6 2D Stress Analysis

### 6.1 Finite element equation formulation

Calculation formula for nodal displacements

$$[k]\{u\} = \{f\} + \{f_t\} + \{f_b\}$$

$$[k] = t \cdot \int_A [B]^T [D] [B] dA$$

$$\{f_t\} = t \cdot \int_A [B]^T [D] \{\epsilon_0\} dA$$

$$\{f_b\} = t \cdot \gamma \cdot \int_A [N]^T [N] dA \cdot \{w\}$$

Calculation formula for element stresses

$$\{\sigma\} = [D]\{\epsilon - \epsilon_0\}$$



In the calculation formula for nodal displacements, the thermal load vector  $\{f_t\}$  is included in the items of load vector. However, in the element stresses calculation, the initial strain due to temperature change shall be subtracted from the strain which are calculated from the nodal displacements shown in above formula.

Equation of relationship between nodal displacement and strain at any points

$$\{\epsilon\} = [B]\{u\}$$

Equation of nodal displacement and displacement at any point

$$\{v\} = [N]\{u\}$$

- $[k]$  : element stiffness matrix
- $\{u\}$  : nodal displacement vector
- $\{f\}$  : nodal external force vector
- $\{f_t\}$  : nodal load vector due to temperature change
- $\{f_b\}$  : nodal body force vector
- $[D]$  : stress-strain relationship matrix
- $t, \gamma$  : element thickness, element unit weight
- $\{w\}$  : nodal acceleration vector (ratio to 'g')
- $\{\epsilon_0\}$  : element strain due to temperature change
- $\{\epsilon\}$  : element strain at any point
- $\{v\}$  : element displacement at any point

## 6.2 Stress-strain relationship on 2D elastic problem

As well known as Hooke's law, stress-strain relationship for 3D isotropic elastic body can be expressed as follow, where  $E$  is elastic modulus,  $\nu$  is Poisson's ratio,  $\alpha$  is thermal expansion coefficient,  $T$  is temperature change.

$$\begin{aligned} \epsilon_x - \alpha T &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] & \epsilon_y - \alpha T &= \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] & \epsilon_z - \alpha T &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} & \gamma_{yz} &= \frac{2(1+\nu)}{E}\tau_{yz} & \gamma_{zx} &= \frac{2(1+\nu)}{E}\tau_{zx} \end{aligned}$$

When considering  $x - y$  plane, in the plane stress state,

$$\sigma_z = 0 \quad \tau_{yz} = 0 \quad \tau_{zx} = 0$$

Therefore,

$$\begin{aligned} \epsilon_x - \alpha T &= \frac{1}{E}(\sigma_x - \nu\sigma_y) & \epsilon_y - \alpha T &= \frac{1}{E}(\sigma_y - \nu\sigma_x) & \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \\ \Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x - \alpha T \\ \epsilon_y - \alpha T \\ \gamma_{xy} \end{Bmatrix} \end{aligned}$$

In the plane strain state,

$$\epsilon_z = 0 \rightarrow \sigma_z = \nu(\sigma_x + \sigma_y) - E\alpha T \quad \tau_{yz} = 0 \quad \tau_{zx} = 0$$

Therefore,

$$\begin{aligned} \epsilon_x - \alpha T &= \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y] & \epsilon_y - \alpha T &= \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x] & \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \\ \Rightarrow \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x - (1+\nu)\alpha T \\ \epsilon_y - (1+\nu)\alpha T \\ \gamma_{xy} \end{Bmatrix} \end{aligned}$$

The rearranged result of above can be shown as follows.

Stress and strain components in 2D elastic problem

$$\text{Stress component: } \{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \text{Strain component: } \{\boldsymbol{\epsilon}\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{Thermal strain component in plane stress state: } \{\boldsymbol{\epsilon}_0\} = \begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$\text{Thermal strain component in plane strain state: } \{\boldsymbol{\epsilon}_0\} = \begin{Bmatrix} (1 + \nu)\alpha T \\ (1 + \nu)\alpha T \\ 0 \end{Bmatrix}$$

where, tensile stress and tensile strain have positive sign, and temperature increase has positive sign.

Stress-strain relationship in 2D elastic problem

$$\text{Plane stress state: } [\mathbf{D}_e] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \quad \begin{array}{l} E : \text{elastic modulus} \\ \nu : \text{Poisson's ratio} \end{array}$$

$$\text{Plane strain state: } [\mathbf{D}_e] = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & 0 \\ \nu & 1 - \nu & 0 \\ 0 & 0 & \frac{1 - 2\nu}{2} \end{bmatrix}$$

### 6.3 Formulation as isoparametric element with 4 nodes 4 Gauss points

#### 6.3.1 Introduction of strain-nodal displacement relationship matrix $[\mathbf{B}]$

The displacements  $u, v$  at any point in a quadrilateral element are assumed as follows, where coordinate  $(a, b)$  is normalized parameter coordinate with range of  $[-1, 1]$  for each of  $a$  or  $b$ ,  $u_{i,j,k,l}$  and  $v_{i,j,k,l}$  are nodal displacements which forms an element.

$$\begin{aligned} u &= N_1(a, b) \cdot u_i + N_2(a, b) \cdot u_j + N_3(a, b) \cdot u_k + N_4(a, b) \cdot u_l \\ v &= N_1(a, b) \cdot v_i + N_2(a, b) \cdot v_j + N_3(a, b) \cdot v_k + N_4(a, b) \cdot v_l \end{aligned}$$

When the displacements at any point in an element is defined as  $\{\mathbf{u}\}$ , and the nodal displacements is defined as  $\{\mathbf{u}_{nd}\}$ , following expression can be obtained using matrix expression.

$$\{\mathbf{u}\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix} = [\mathbf{N}]\{\mathbf{u}_{nd}\}$$

The strain at any point in an element  $\{\boldsymbol{\epsilon}\}$  can be expressed using nodal displacement  $\{\mathbf{u}_{nd}\}$ .

$$\{\boldsymbol{\epsilon}\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \\ u_l \\ v_l \end{Bmatrix} = [\mathbf{B}]\{\mathbf{u}_{nd}\}$$

### 6.3.2 Element stiffness matrix

A stiffness matrix of 4 nodes isoparametric element  $[k]$  can be expressed as follow using the constant element thickness  $t$  and stress-strain relationship matrix  $[D_e]$ .

$$[k] = t \int_{-1}^1 \int_{-1}^1 [B]^T [D_e] [B] \cdot \det(J) \cdot da \cdot db = t \cdot \sum_{kk=1}^4 \{ [B]^T [D_e] [B] \cdot \det(J) \}_{kk}$$

### 6.3.3 Nodal force vector due to temperature change

Nodal force vector  $\{f_t\}$  due to thermal strain  $\{\epsilon_0\}$  can be expressed as follow.

$$\{f_t\} = t \int_{-1}^1 \int_{-1}^1 [B]^T [D_e] \{\epsilon_0\} \cdot \det(J) \cdot da \cdot db = t \cdot \sum_{kk=1}^4 \{ [B]^T [D_e] \{\epsilon_0\} \cdot \det(J) \}_{kk}$$

The component of strain due to temperature change is shown below.

$$T_p = N_1 \cdot \phi_i + N_2 \cdot \phi_j + N_3 \cdot \phi_k + N_4 \cdot \phi_l$$

$$\text{Plane stress state: } \{\epsilon_0\} = \begin{Bmatrix} \alpha T_p \\ \alpha T_p \\ 0 \end{Bmatrix}$$

$$\text{Plane strain state: } \{\epsilon_0\} = \begin{Bmatrix} (1 + \nu) \alpha T_p \\ (1 + \nu) \alpha T_p \\ 0 \end{Bmatrix}$$

where,  $\alpha$  is thermal expansion coefficient,  $\nu$  is Poisson's ratio,  $T_p$  is temperature change at any point in an element,  $\{\phi_i \ \phi_j \ \phi_k \ \phi_l\}^T$  is nodal temperature change vector,  $[N_1 \ N_2 \ N_3 \ N_4]$  is elements of interpolation function  $[N]$ . And temperature increase has positive sign.

### 6.3.4 Nodal force vector due to body force

Body force (inertia force) at any point in a element  $\{F\} = \{F_{B_x}, F_{B_y}\}^T$  can be expressed using nodal force vector  $\{w\}$  and interpolation function  $[N]$ .

$$\{F\} = [N] \{w\}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$\{w\} = \gamma \cdot \{k_h \ k_v \ k_h \ k_v \ k_h \ k_v \ k_h \ k_v\}^T$$

where,  $\gamma$  is element unit weight (mass  $\times$  gravity acceleration),  $k_h$  and  $k_v$  are horiaontal and vertical acceleration (ratio to 'g'). From above, nodal force vector due to body force (inertia force)  $\{f_b\}$  becomes can be calculated as follow.

$$\{f_b\} = t \cdot \int_A [N]^T \{F\} dA = t \cdot \gamma \cdot \int_A [N]^T [N] dA \cdot \{w'\} = t \cdot \gamma \cdot \sum_{kk=1}^4 \{ [N]^T [N] \cdot \det(J) \}_{kk} \cdot \{w'\}$$

The vector  $\{w\}$  is defined as follow by pulling out  $\gamma$ .

$$\{w'\} = \{k_h \ k_v \ k_h \ k_v \ k_h \ k_v \ k_h \ k_v\}^T$$

## 7 Axisymmetric Stress Analysis

### 7.1 Stress-strain relationship and strain-displacement relationship in axisymmetric problem

Regarding the coordinate, it is defined that the direction of rotation axis is set to z-direction, the radius direction is set to r-direction and circumferential direction is set to  $\theta$ -direction. Using this coordinate system, stress-strain relationship in axisymmetric problem is shown below.

$$\begin{cases} \epsilon_z - \alpha T = \frac{1}{E} \{ \sigma_z - \nu(\sigma_r + \sigma_\theta) \} \\ \epsilon_r - \alpha T = \frac{1}{E} \{ \sigma_r - \nu(\sigma_z + \sigma_\theta) \} \\ \epsilon_\theta - \alpha T = \frac{1}{E} \{ \sigma_\theta - \nu(\sigma_z + \sigma_r) \} \\ \gamma_{zr} = \frac{2(1+\nu)}{E} \cdot \tau_{zr} \end{cases}$$

where  $E$  is elastic modulus,  $\nu$  is Poisson's ratio,  $\alpha$  is thermal expansion coefficient,  $T$  is temperature change. By transforming above equations, followings can be obtained.

$$\begin{cases} \sigma_z = \frac{E}{(1+\nu)(1-2\nu)} \{ (1-\nu)\epsilon_z + \nu\epsilon_r + \nu\epsilon_\theta - (1+\nu)\alpha T \} \\ \sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \{ \nu\epsilon_z + (1-\nu)\epsilon_r + \nu\epsilon_\theta - (1+\nu)\alpha T \} \\ \sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)} \{ \nu\epsilon_z + \nu\epsilon_r + (1-\nu)\epsilon_\theta - (1+\nu)\alpha T \} \\ \tau_{zr} = \frac{E}{2(1+\nu)} \cdot \gamma_{zr} \end{cases}$$

Using matrix and vector format, above can be expressed as follows.

$$\{\boldsymbol{\sigma}\} = [\mathbf{D}_e] \{\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_0\}$$

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_z \\ \sigma_r \\ \sigma_\theta \\ \tau_{zr} \end{Bmatrix} \quad \{\boldsymbol{\epsilon}\} = \begin{Bmatrix} \epsilon_z \\ \epsilon_r \\ \epsilon_\theta \\ \gamma_{zr} \end{Bmatrix} \quad \{\boldsymbol{\epsilon}_0\} = \begin{Bmatrix} \alpha T \\ \alpha T \\ \alpha T \\ 0 \end{Bmatrix}$$

$$[\mathbf{D}_e] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Strain-displacement relationship is shown below, where  $w$  is the displacement in rotation axis direction and  $u$  is the displacement in radius direction.

$$\{\boldsymbol{\epsilon}\} = \begin{Bmatrix} \epsilon_z \\ \epsilon_r \\ \epsilon_\theta \\ \gamma_{zr} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \end{Bmatrix}$$

## 7.2 Formulation as isoparametric element with 4 nodes 4 Gauss points

### 7.2.1 Introduction of strain-nodal displacement relationship matrix $[\mathbf{B}]$

The displacements in rotation axis direction  $w$  and the displacement in radial direction  $u$  at any point in a axisymmetric quadrilateral element are assumed as follows, where coordinate  $(a,b)$  is normalized parameter coordinate with range of  $[-1,1]$  for each of  $a$  or  $b$ ,  $w_{i,j,k,l}$  and  $u_{i,j,k,l}$  are nodal displacements which forms an element.

$$\begin{aligned} w &= N_1(a,b) \cdot w_i + N_2(a,b) \cdot w_j + N_3(a,b) \cdot w_k + N_4(a,b) \cdot w_l \\ u &= N_1(a,b) \cdot u_i + N_2(a,b) \cdot u_j + N_3(a,b) \cdot u_k + N_4(a,b) \cdot u_l \end{aligned}$$

When the displacements at any point in an element is defined as  $\{\mathbf{u}\}$ , and the nodal displacements is defined as  $\{\mathbf{u}_{nd}\}$ , following expression can be obtained using matrix expression.

$$\{\mathbf{u}\} = \begin{Bmatrix} w \\ u \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} w_i \\ u_i \\ w_j \\ u_j \\ w_k \\ u_k \\ w_l \\ u_l \end{Bmatrix} = [N]\{\mathbf{u}_{nd}\}$$

The strain at any point in an element  $\{\epsilon\}$  can be expressed as follow using nodal displacement  $\{\mathbf{u}_{nd}\}$ .

$$\{\epsilon\} = \begin{Bmatrix} \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} \\ \frac{\partial r}{r} \\ \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_4}{\partial z} & 0 \\ 0 & \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} \\ 0 & \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 & \frac{N_4}{r} \\ \frac{\partial N_1}{\partial r} & \frac{r}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{r}{\partial z} & \frac{\partial N_3}{\partial r} & \frac{r}{\partial z} & \frac{\partial N_4}{\partial r} & \frac{r}{\partial z} \end{bmatrix} \begin{Bmatrix} w_i \\ u_i \\ w_j \\ u_j \\ w_k \\ u_k \\ w_l \\ u_l \end{Bmatrix} = [B]\{\mathbf{u}_{nd}\}$$

The value of  $r$  coordinate at any point in an element is evaluated using following equation.

$$r = N_1 \cdot r_i + N_2 \cdot r_j + N_3 \cdot r_k + N_4 \cdot r_l$$

where,  $r_i, r_j, r_k, r_l$  are the value of  $r$  coordinates which forms an element.

## 7.2.2 Element stiffness matrix

When an axisymmetric finite element is considered, the volume of small element which is located at  $r$  in radial direction from the rotation center becomes  $r \times d\theta \times dr \times dz$ . When integrals and load vector is evaluated with angle of 1 radian, the result of an integral variable conversion for Gauss-Legendre quadrature can be shown as follow.

$$r \times d\theta \times dr \times dz = r \times dr \times dz = r \times \det(J) \times da \times db$$

As a result, element stiffness matrix of axisymmetric element  $[k]$  can be expressed as follow using stress-strain relationship matrix  $[D_e]$ .

$$[k] = \int_{-1}^1 \int_{-1}^1 [B]^T [D_e] [B] \cdot r \cdot \det(J) \cdot da \cdot db = \sum_{kk=1}^4 \{[B]^T [D_e] [B] \cdot r \cdot \det(J)\}_{kk}$$

# 8 2D Saturated-Unsaturated Seepage Flow Analysis

## 8.1 Finite element equation of 2D steady saturated seepage flow problem

Governing equation of steady saturated seepage flow problem with 2D orthotropic material followed Darcy's law is shown below in  $x - y$  rectangular coordinate system, where the depth is not considered.

$$k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\phi$ : total head     $k_x$ : permeability in  $x$ -direction     $k_y$ : permeability in  $y$ -direction

To define the unknown  $\phi$  which satisfies following weak formulation of governing equation for any  $\delta\phi$  is considered using Galerkin method.

$$\int_A \delta\phi \left( k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} \right) dA = 0$$

When the total head  $\phi$  at any point can be expressed as follow using interpolation function matrix  $[N]$  and nodal total head vector  $\{\mathbf{h}\}$ ,

$$\phi(x, y) = [N(x, y)]\{\mathbf{h}\}$$

the weak formulation of governing equation can be expressed as follow.

$$\{\delta \mathbf{h}\}^T \int_A [\mathbf{N}]^T \left( k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} \right) dA = 0$$

At this time, the surface integral value shall be zero, because  $\{\delta \mathbf{h}\}$  is arbitrary any nodal total headvector. Next, using a partial integration formula

$$f \cdot g' = (f \cdot g)' - f' \cdot g$$

and Green's theorem

$$\iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \int_C \left( P \frac{\partial x}{\partial n} + Q \frac{\partial y}{\partial n} \right) ds$$

following equation can be derived.

$$\begin{aligned} 0 &= \int_A [\mathbf{N}]^T \left( k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} \right) dA \\ &= \int_A \left\{ k_x \left( \frac{\partial \left( [\mathbf{N}]^T \frac{\partial \phi}{\partial x} \right)}{\partial x} - \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial \phi}{\partial x} \right) + k_y \left( \frac{\partial \left( [\mathbf{N}]^T \frac{\partial \phi}{\partial y} \right)}{\partial y} - \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial \phi}{\partial y} \right) \right\} dA \\ &= \int_s [\mathbf{N}]^T \left( k_x \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial n} + k_y \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial n} \right) ds - \int_A \left( k_x \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial [\mathbf{N}]}{\partial x} + k_y \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial [\mathbf{N}]}{\partial y} \right) dA \cdot \{\mathbf{h}\} \end{aligned}$$

And when the velocity in  $x$  and  $y$ -directions are defined as  $v_x$  and  $v_y$ , following equations can be obtained using Darcy's law.

$$v_x = -k_x \frac{\partial \phi}{\partial x} \quad v_y = -k_y \frac{\partial \phi}{\partial y}$$

As a result, element finite element equation can be obtained as follow.

$$\begin{aligned} [\mathbf{k}]\{\mathbf{h}\} &= \{\mathbf{q}\} \\ [\mathbf{k}] &= \int_A \left( k_x \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial [\mathbf{N}]}{\partial x} + k_y \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial [\mathbf{N}]}{\partial y} \right) dA \\ \{\mathbf{q}\} &= - \int_s [\mathbf{N}]^T \left( v_x \frac{\partial x}{\partial n} + v_y \frac{\partial y}{\partial n} \right) ds \end{aligned}$$

$[\mathbf{k}]$ : permeability matrix     $\{\mathbf{h}\}$ : nodal total head vector     $\{\mathbf{q}\}$ : nodal discharge vector

where,  $\{\mathbf{q}\}$  is integral value of velocity vector along the side, and it is easy to understand that it means discharge.

## 8.2 Formulation as isoparametric element with 4 nodes 4 Gauss points

### 8.2.1 Permeability matrix

When an isoparametric element has 4 Gauss integral points, an element permeability matrix can be expressed as follow.

$$\begin{aligned} [\mathbf{k}] &= \int_A \left( k_x \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial [\mathbf{N}]}{\partial x} + k_y \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial [\mathbf{N}]}{\partial y} \right) dA \\ &= \int_{-1}^1 \int_{-1}^1 \left( k_x \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial [\mathbf{N}]}{\partial x} + k_y \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial [\mathbf{N}]}{\partial y} \right) \cdot \det(J) \cdot da \cdot db \\ &= \sum_{kk=1}^4 \left( k_x \frac{\partial [\mathbf{N}]^T}{\partial x} \frac{\partial [\mathbf{N}]}{\partial x} + k_y \frac{\partial [\mathbf{N}]^T}{\partial y} \frac{\partial [\mathbf{N}]}{\partial y} \right) \cdot \det(J) \end{aligned}$$

Regarding nodal discharge vector  $\{\mathbf{q}\}$ , it can be inputted as a equivalent nodal discharge vector directly.

### 8.2.2 Average velocity in an element

The total head at any point in an element  $\phi$  can be expressed as follow using interpolation function  $[N]$  and nodal total head vector  $\{h\}$ .

$$\phi = [N]\{h\}$$

Therefore, the mean velocity at any point in an element  $v_x$  and  $v_y$  can be expressed as follows using Darcy's law.

$$v_x = -k_x \frac{\partial \phi}{\partial x} = -k_x \frac{\partial [N]}{\partial x} \{h\}$$

$$v_y = -k_y \frac{\partial \phi}{\partial y} = -k_y \frac{\partial [N]}{\partial y} \{h\}$$

In actual calculation, following formula can be used to obtain the average velocity in an element.

$$v_x = -\frac{k_x}{4} \sum_{kk=1}^4 \left( \frac{\partial [N]}{\partial x} \{h\} \right)$$

$$v_y = -\frac{k_y}{4} \sum_{kk=1}^4 \left( \frac{\partial [N]}{\partial y} \{h\} \right)$$

(Notice) Above discussion is for orthotropic material. However, a program introduced can treat only isotropic material.

## 8.3 Discussion of extension to unsaturated seepage flow analysis

### 8.3.1 Saturated seepage flow analysis

A saturated seepage flow anaalysis is a problem to solve following simultaneous linear equations.

$$[k]\{h\} = \{q\}$$

$$\begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & k_{22} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} h_{\text{unknown}} \\ h_{\text{unknown}} \\ \dots \\ h_{\text{given}} \end{Bmatrix} = \begin{Bmatrix} q_{\text{given}} \\ q_{\text{given}} \\ \dots \\ q_{\text{unknown}} \end{Bmatrix}$$

where,  $\{q\}$  is nodal discharge vector,  $\{h\}$  is total head vector,  $[k]$  is permeability matrix, and they have following characteristics.

- Permeability matrix  $[k]$  is a constant.
- components of nodal total head vector  $\{h\}$  which are corresponding to known components of nodal discharge vector  $\{q\}$  are unknown variavles.
- components of nodal total head vector  $\{h\}$  which are corresponding to unknown components of nodal discharge vector  $\{q\}$  are known variavles.

Therefore, after treatment of the relationship between known and unknown variables, the solution can be obtained by only one time calculation.

### 8.3.2 Saturated-unsaturated seepage flow analysis

simultaneous equations in saturated-unsaturated seepage analysis have following characteristics.

- Three types boundary conditions shall be considered such as given discharge, given total head and seepage face which have the possibilities of occurance of free surface.
- Either discharge or total head becomes known variable except seepage face boundary.
- Both of discharge and total head become unknown variables on the seepage face boundary.
- However, seepage face boundary has only outflow condition, because it touches to the air. Inaddition, pressure head on that boundary should be less than or equal to zero.
- The permeability matrix in the saturated area is constant. However, the permeability matrix in the unsaturated area becomes a function of suction head (negative pressure head). Therefore, simultaneous equations shall be solved by iterative calculation.

As a iterative calculation method, successive substitution method is applied. In this method, initial total head are given for all nodes, and ibtained solutions are used as input values for next iterative calculation. Regarding the values of initial total heads, it is better to give the minimum value of z-coordinate of the model (minimum altitude in the model).

### 8.3.3 Unsaturated permeability characteristics (van Genuchten model)

As unsaturated permeability characteristics, van Genuchten model shown below is applied, because it is convenient for programming to treat continuous function as material characteristics.

$$S_e = \left\{ \frac{1}{1 + (\alpha \cdot h_s)^n} \right\}^m \quad n = \frac{1}{1 - m} \quad (0 < m < 1)$$

$$K_r = (S_e)^{0.5} \cdot \left\{ 1 - \left( 1 - S_e^{1/m} \right)^m \right\}^2 \quad (0 \leq S_r, K_r \leq 1)$$

$$K = K_r \cdot K_0$$

$S_e$	Degree of saturation
$K_r$	Relative hydraulic conductivity function
$h_s$	Suction head (positive sign)
$\alpha$	Scaling parameter
$m$	Non-dimensional parameter
$K$	Permiability coefficient
$K_0$	Saturated permiability coefficient

## 9 2D Thermal Conductivity Analysis

### 9.1 Finite element equation of 2D unsteady thermal conductivity problem

Governing equation of 2D unsteady thermal conductivity problem with 2D isotropic material is shown below in  $x - y$  rectangular coordinate system, where the depth is not considered.

$$\rho c \frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{Q}$$

$T$	: temperature	$t$	: time	$\dot{Q}$	: heat rate
$\rho$	: density	$c$	: specific heat	$\kappa$	: heat conductivity coefficient

Heat flux at boundary  $q$  can be expressed as follow using Fourier's law. where  $n$  is an outward normal on the boundary.

$$q = -\kappa \frac{\partial T}{\partial n}$$

To define the unknown  $T$  which satisfies following weak formulation of governing equation for any  $\delta T$  is considered using Galerkin method.

$$\int_A \delta T \left\{ \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{Q} - \rho c \frac{\partial T}{\partial t} \right\} dA = 0$$

When the temperature at any point  $T$  can be expressed as follow using interpolation function matrix  $[N]$  and nodal temperature vector  $\{\phi\}$ ,

$$T(x, y, t) = [N(x, y)]\{\phi(t)\}$$

the weak formulation of governing equation can be expressed as follow.

$$\{\delta\phi\}^T \int_A [N]^T \left\{ \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{Q} - \rho c \frac{\partial T}{\partial t} \right\} dA = 0$$

At this time, the surface integral value shall be zero, because  $\{\delta\phi\}$  is arbitrary nodal total head vector. Next, using a partial integration formula

$$f \cdot g' = (f \cdot g)' - f' \cdot g$$

and Green's theorem

$$\iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \int_C \left( P \frac{\partial x}{\partial n} + Q \frac{\partial y}{\partial n} \right) ds$$

and using Fourier's law, the item related to  $\kappa$  becomes shown below.



$$\begin{aligned}
& \int_A [N]^T \left\{ \kappa \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \right\} dA \\
&= \int_A \kappa \left\{ \left( \frac{\partial \left( [N]^T \frac{\partial h}{\partial x} \right)}{\partial x} - \frac{\partial [N]^T}{\partial x} \frac{\partial h}{\partial x} \right) + \left( \frac{\partial \left( [N]^T \frac{\partial h}{\partial y} \right)}{\partial y} - \frac{\partial [N]^T}{\partial y} \frac{\partial h}{\partial y} \right) \right\} dA \\
&= \int_s [N]^T \kappa \left( \frac{\partial h}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial x} \right) ds - \int_A \kappa \left( \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA \cdot \{\phi\} \\
&= - \int_s q [N]^T ds - \int_A \kappa \left( \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA \cdot \{\phi\}
\end{aligned}$$

And following equation can be obtained.

$$\int_A [N]^T \left( \dot{Q} - \rho c \frac{\partial T}{\partial t} \right) dA = \int_A \dot{Q} [N]^T dA - \int_A \rho c [N]^T [N] dA \cdot \left\{ \frac{\partial \phi}{\partial t} \right\}$$

As a result, element finite element equation can be obtained as follow.

$$\begin{aligned}
[k] \{\phi\} + [c] \left\{ \frac{\partial \phi}{\partial t} \right\} &= \{f\} \\
[k] &= \int_A \kappa \left( \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA \\
[c] &= \int_A \rho c [N]^T [N] dA \\
\{f\} &= \int_A \dot{Q} [N]^T dA - \int_s q [N]^T ds
\end{aligned}$$

$[k]$  : heat conduction matrix       $[c]$  : heat capacity matrix  
 $\{\phi\}$  : nodal temperature vector       $\{f\}$  : nodal heat flux vector

### 9.1.1 Explanation about boundary condition

In case that heat flux  $q_0$  is given on the boundary (side of element)

$$\int_s q [N]^T ds = \int_s q_0 [N]^T ds$$

for adiabatic boundary, it becomes  $q_0 = 0$ . In case that heat transfer boundary (side of element), using heat transfer rate  $\alpha_c$  and outside temperature  $T_c$ ,

$$\begin{aligned}
\int_s q [N]^T ds &= \int_s \alpha_c (T - T_c) [N]^T ds \\
&= \int_s \alpha_c [N]^T [N] ds \cdot \{\phi\} - \int_s \alpha_c T_c [N]^T ds
\end{aligned}$$

Above items shall be added to heat conduction matrix or heat flux vector.

### 9.1.2 Finite element equation considered boundary condition effects

$$[k] \{\phi\} + [c] \left\{ \frac{\partial \phi}{\partial t} \right\} = \{f\}$$

$$\begin{aligned}
[k] &= \int_A \kappa \left( \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA + \int_s \alpha_c [N]^T [N] ds \\
[c] &= \int_A \rho c [N]^T [N] dA \\
\{f\} &= \int_A \dot{Q} [N]^T dA + \int_s \alpha_c T_c [N]^T ds
\end{aligned}$$

$[k]$	: heat conduction matrix. 2nd item is due to effect of heat transfer boundary
$[c]$	: heat capacity matrix
$\{\phi\}$	: nodal temperature vector
$\{f\}$	: nodal heat flux vector. 1st item is due to effect of heat rate, 2nd item is due to effect of heat transfer boundary
$[N]$	: interpolation function matrix
$\kappa, \rho, c$	: heat conductivity coefficient, density, specific heat
$\alpha_c, T_c$	: heat transfer rate, outside temperature of heat transfer boundary
$\dot{Q}$	: heat rate
$\int_A, \int_S$	: surface integral of element area, curvilinear integral along the side

### 9.1.3 Explanation of heating material

Treatment of Heating material such as cement concrete is considered. The adiabatic temperature rise is assumed as follow.

$$T = K \cdot (1 - e^{-\alpha \cdot t})$$

$T$ : adiabatic temperature rise                       $K$ : maximum temperature rise  
 $\alpha$ : parameter for heat generation rate       $t$ : time

$K$  (Tk) and  $\alpha$  (Al) in above equation will be inputted as material characteristics. Using above, heat value  $Q$  and heat rate  $\dot{Q}$  can be expressed as follow.

$$Q = \rho \cdot c \cdot T(t) \quad \rightarrow \quad \dot{Q} = \rho \cdot c \cdot T_k \cdot \alpha \cdot e^{-\alpha t}$$

### 9.1.4 Solution of unsteady finite element equation

Unsteady finite element equation can be solved using Crank-Nicolson method. The finite element equation of analysis model is expressed as follow.

$$[K]\{\Phi\} + [C] \left\{ \frac{\partial \Phi}{\partial t} \right\} = \{F\}$$

where, nodal temperature vector  $\{\Phi\}$ , deviation of nodal temperature vector  $\{\partial \Phi / \partial t\}$  and heat flux vector  $\{F\}$  at the time  $t + \Delta t / 2$  are defined as follows.

$$\begin{aligned} \left\{ \Phi \left( t + \frac{\Delta}{2} \right) \right\} &= \frac{\{\Phi(t + \Delta t)\} + \{\Phi(t)\}}{2} \\ \left\{ \frac{\partial \Phi}{\partial t} \left( t + \frac{\Delta}{2} \right) \right\} &= \frac{\{\Phi(t + \Delta t)\} - \{\Phi(t)\}}{\Delta t} \\ \left\{ F \left( t + \frac{\Delta}{2} \right) \right\} &= \frac{\{F(t + \Delta t)\} + \{F(t)\}}{2} \end{aligned}$$

By substituting above relationship to the finite element equation, following equation can be obtained. As a result, unknown nodal temperature at time  $t + \Delta t$  can be calculated using known temperature at time  $t$ .

$$\left( \frac{1}{2}[K] + \frac{1}{\Delta t}[C] \right) \{\Phi(t + \Delta t)\} = \left( -\frac{1}{2}[K] + \frac{1}{\Delta t}[C] \right) \{\Phi(t)\} + \frac{\{F(t + \Delta t)\} + \{F(t)\}}{2}$$

If material considered does not have the time dependence, it is possible to calculate the temperature at next step by only one time calculation of inverse matrix. For the calculation of inverse matrix, `numpy.linalg.inv(A)` in Python can be used.

## 9.2 Formulation as isoparametric element with 4 nodes 4 Gauss points

The matrices and vectors can be expressed as follows for an isoparametric element with 4 nodes 4 Gauss points, where  $\ell$  is the length of the side with specified boundary condition of an element.



Next, the linearization of  $\ddot{u}$  is considered as follow.

$$\ddot{u}(t) = \frac{\ddot{u}(t + \Delta t) - \ddot{u}(t)}{\Delta t}$$

Using above relationship, following equations can be obtained.

$$u(t + \Delta t) = u(t) + \Delta t \cdot \dot{u}(t) + \left(\frac{1}{2} - \beta\right) (\Delta t)^2 \cdot \ddot{u}(t) + \beta (\Delta t)^2 \cdot \ddot{u}(t + \Delta t)$$

$$\dot{u}(t + \Delta t) = \dot{u}(t) + (1 - \gamma) \cdot \Delta t \cdot \ddot{u}(t) + \gamma \cdot \Delta t \cdot \ddot{u}(t + \Delta t)$$

When the application of the equations to finite element method, it is better to solve the equation with unknown displacement. So, the equations can be rearranged as follows.

$$\ddot{u}(t + \Delta t) = \frac{1}{\beta(\Delta t)^2} \cdot [u(t + \Delta t) - u(t)] - \frac{1}{\beta\Delta t} \cdot \dot{u}(t) - \left(\frac{1}{2\beta} - 1\right) \cdot \ddot{u}(t)$$

$$\dot{u}(t + \Delta t) = \frac{\gamma}{\beta\Delta t} \cdot [u(t + \Delta t) - u(t)] - \left(\frac{\gamma}{\beta} - 1\right) \cdot \dot{u}(t) - \left(\frac{\gamma}{2\beta} - 1\right) \Delta t \cdot \ddot{u}(t)$$

Substituting above to the equation of motion, followings can be obtained.

$$\left(\frac{1}{\beta(\Delta t)^2} \cdot m + \frac{\gamma}{\beta\Delta t} \cdot c + k\right) \cdot u(t + \Delta t) = f(t + \Delta t)$$

$$+ m \cdot \left[\frac{1}{\beta(\Delta t)^2} \cdot u(t) + \frac{1}{\beta\Delta t} \cdot \dot{u}(t) + \left(\frac{1}{2\beta} - 1\right) \cdot \ddot{u}(t)\right]$$

$$+ c \cdot \left[\frac{\gamma}{\beta\Delta t} \cdot u(t) + \left(\frac{\gamma}{\beta} - 1\right) \cdot \dot{u}(t) + \left(\frac{\gamma}{2\beta} - 1\right) \Delta t \cdot \ddot{u}(t)\right]$$

Usually,  $\gamma = 1/2$  is applied, and the case of  $\beta = 1/4$  means the average acceleration method, the case of  $\beta = 1/6$  means the linear acceleration method.

## 10.2 The equation to be solved with matrix expression

The equations for SDOF can be changed with matrix expressed as follows by fixing  $\gamma = 1/2$ .

### 10.2.1 Calculation formula of displacement

$$\left(\frac{1}{\beta(\Delta t)^2} [\mathbf{m}] + \frac{1}{2\beta\Delta t} [\mathbf{c}] + [\mathbf{k}]\right) \{\mathbf{u}(t + \Delta t)\} = \{\mathbf{f}(t + \Delta t)\} + [\mathbf{m}]\{\mathbf{w}_a(t)\} + [\mathbf{c}]\{\mathbf{w}_b(t)\}$$

$$\{\mathbf{w}_a(t)\} = \frac{1}{\beta(\Delta t)^2} \{\mathbf{u}(t)\} + \frac{1}{\beta\Delta t} \{\dot{\mathbf{u}}(t)\} + \left(\frac{1}{2\beta} - 1\right) \{\ddot{\mathbf{u}}(t)\}$$

$$\{\mathbf{w}_b(t)\} = \frac{1}{2\beta\Delta t} \{\mathbf{u}(t)\} + \left(\frac{1}{2\beta} - 1\right) \{\dot{\mathbf{u}}(t)\} + \left(\frac{1}{4\beta} - 1\right) \Delta t \{\ddot{\mathbf{u}}(t)\}$$

### 10.2.2 Calculation formulas of velocity and acceleration

After calculation of the displacement  $\{\mathbf{u}(t + \Delta t)\}$ , the velocity  $\{\dot{\mathbf{u}}(t + \Delta t)\}$  and the acceleration  $\{\ddot{\mathbf{u}}(t + \Delta t)\}$  can be calculated using below equations.

$$\{\dot{\mathbf{u}}(t + \Delta t)\} = \frac{1}{2\beta\Delta t} (\{\mathbf{u}(t + \Delta t)\} - \{\mathbf{u}(t)\}) - \left(\frac{1}{2\beta} - 1\right) \{\dot{\mathbf{u}}(t)\} - \left(\frac{1}{4\beta} - 1\right) \Delta t \{\ddot{\mathbf{u}}(t)\}$$

$$\{\ddot{\mathbf{u}}(t + \Delta t)\} = \frac{1}{\beta(\Delta t)^2} (\{\mathbf{u}(t + \Delta t)\} - \{\mathbf{u}(t)\}) - \frac{1}{\beta\Delta t} \{\dot{\mathbf{u}}(t)\} - \left(\frac{1}{2\beta} - 1\right) \{\ddot{\mathbf{u}}(t)\}$$

$[\mathbf{m}]$  : mass matrix                       $\{\ddot{\mathbf{u}}\}$  : acceleration vector  
 $[\mathbf{c}]$  : damping matrix                     $\{\dot{\mathbf{u}}\}$  : velocity vector  
 $[\mathbf{k}]$  : stiffness matrix                     $\{\mathbf{u}\}$  : displacement vector  
 $\{\mathbf{f}\}$  : external force vector

where,  $\beta = 1/4$  means average acceleration method,  $\beta = 1/6$  means linear acceleration method.

### 10.3 Stiffness matrix and mass matrix

$$\begin{aligned} \{f_*\} &= [k]\{u_*\} \\ \{f_*\} &= \{N_i \quad S_i \quad M_i \quad N_j \quad S_j \quad M_j\}^T \\ \{u_*\} &= \{u_i \quad v_i \quad \theta_i \quad u_j \quad v_j \quad \theta_j\}^T \\ [k] &= \begin{bmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 & 6EI/\ell^2 & 0 & -12EI/\ell^3 & 6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 4EI/\ell & 0 & -6EI/\ell^2 & 2EI/\ell \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 & -6EI/\ell^2 & 0 & 12EI/\ell^3 & -6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 2EI/\ell & 0 & -6EI/\ell^2 & 4EI/\ell \end{bmatrix} \\ [m] &= \frac{\gamma \cdot A \cdot \ell}{g} \begin{bmatrix} 1/3 & 0 & 0 & 1/6 & 0 & 0 \\ 0 & 13/35 & 11\ell/210 & 0 & 9/70 & -13\ell/420 \\ 0 & 11\ell/210 & \ell^2/105 & 0 & 13\ell/420 & -\ell^2/140 \\ 1/6 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 9/70 & 13\ell/420 & 0 & 13/35 & -11\ell/210 \\ 0 & -13\ell/420 & -\ell^2/140 & 0 & -11\ell/210 & \ell^2/105 \end{bmatrix} \\ [T] &= \begin{bmatrix} \cos \phi & \sin \phi & 0 & 0 & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \phi & \sin \phi & 0 \\ 0 & 0 & 0 & -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\{f_*\}$ : element nodal force vector       $E$ : elastic modulus of an element  
 $\{u_*\}$ : element nodal displacement vector       $A$ : section area of an element  
 $[k]$ : element stiffness matrix       $I$ : moment of inertia of an element  
 $[m]$ : element mass matrix       $\ell$ : length of an element  
 $[T]$ : coordinate transformation matrix       $\gamma$ : unit weight of an element  
 $[T]^T$ : translocation of matrix  $[T]$        $g$ : gravity acceleration

The coordinate transformation matrix  $[T]$  transforms the displacements or forces in global coordinate system to those in local coordinate system. The relationship between the stiffness matrix in global coordinate system  $[K]$  and that in local coordinate system  $[k]$  is shown below.

$$[K] = [T]^T [k] [T]$$

The mass matrix in global coordinate system  $[M]$  and that in local coordinate system  $[m]$  have the same relationship as the stiffness matrices shown below.

$$[M] = [T]^T [m] [T]$$

### 10.4 Damping matrix

In the actual calculation, a damping matrix  $[C]$  should be defined to consider the effect of damping of the structure. In this section, definition of the damping matrix is discussed. For simplification, the equation of motion of SDOF system is considered.

$$m \cdot \ddot{u} + c \cdot \dot{u} + k \cdot u = -m \cdot \ddot{\phi}$$

$$\ddot{u} + \frac{c}{m} \cdot \dot{u} + \frac{k}{m} \cdot u = -\ddot{\phi}$$

$$\ddot{u} + 2 \cdot h \cdot \omega \cdot \dot{u} + \omega^2 \cdot u = -\ddot{\phi}$$

$$h = \frac{c/m}{2 \cdot \omega} \quad \omega = \sqrt{\frac{k}{m}}$$

Hear, Rayleigh damping is considered. In the theorem of Rayleigh damping, damping coefficient is proportional to the mass and stiffness coefficient. This relationship can be expressed as follow.

$$c = \zeta_m \cdot m + \zeta_k \cdot k$$

Considering above, damping ratio  $h$  can be rearranged as follow, where  $\omega$  is fundamental circular frequency,  $f$  is natural frequency.

$$h = \frac{c/m}{2 \cdot \omega} = \frac{\zeta_m \cdot m + \zeta_k \cdot k}{2 \cdot \omega \cdot m} = \frac{\zeta_m}{2 \cdot \omega} + \frac{\zeta_k \cdot \omega}{2} = \frac{1}{4\pi f} \cdot \zeta_m + \pi f \cdot \zeta_k$$

Accordingly, if damping rates  $h_i, h_j$  and natural frequency  $f_i, f_j$  of  $i$ th mode and  $j$ th mode, proportional constants  $\zeta_m$  and  $\zeta_k$  can be obtained by satisfying above equations simultaneously as shown below.

$$\begin{cases} h_i = \frac{1}{4\pi f_i} \cdot \zeta_m + \pi f_i \cdot \zeta_k \\ h_j = \frac{1}{4\pi f_j} \cdot \zeta_m + \pi f_j \cdot \zeta_k \end{cases} \implies \begin{cases} \zeta_m = \frac{4\pi \cdot f_i \cdot f_j \cdot (f_j \cdot h_i - f_i \cdot h_j)}{f_j^2 - f_i^2} \\ \zeta_k = \frac{f_j \cdot h_j - f_i \cdot h_i}{\pi(f_j^2 - f_i^2)} \end{cases}$$

The matrix expression of damping coefficient is shown below using two scalar constants  $\zeta_m$  and  $\zeta_k$ .

$$[\mathbf{C}] = \zeta_m \cdot [\mathbf{M}] + \zeta_k \cdot [\mathbf{K}]$$

As a method to define the constants of Rayleigh damping, following method can be used.

- To set damping rate as a popular value such as  $h_1 = h_2 = 0.05$ .
- To obtain the natural frequencies  $f_1$  and  $f_2$  for 1st and 2nd mode using eigenvalue analysis.
- To define  $\zeta_m$  and  $\zeta_k$  using two values of natural frequency ( $f_1, f_2$ ) and two values of damping rate ( $h_1 = h_2 = 0.05$ )

## 10.5 Eigenvalue analysis

The problem to obtain the fundamental circular frequency is a problem to solve the characteristic equation shown below as a generalized eigenvalue problem.

$$([\mathbf{K}] - \omega^2[\mathbf{M}])\{U\} = \{0\}$$

After getting the fundamental circular frequency  $\omega_n$ , natural frequency  $f_n$  can be obtained using following equation.

$$f_n = \frac{\omega_n}{2\pi}$$

## 11 2D Frame Geometrically Nonlinear Analysis

### 11.1 Incremental stiffness equation

As well known, an incremental stiffness equation including geometrical nonlinear terms can be expressed shown below:

$$\{\Delta f\} = [K_T]\{\Delta u\} \quad [K_T] = [K_L] + [K_G]$$

$$\{\Delta f\} = \{\Delta N_i \quad \Delta S_i \quad \Delta M_i \quad \Delta N_j \quad \Delta S_j \quad \Delta M_j\}^T$$

$$\{\Delta u\} = \{\Delta u_i \quad \Delta v_i \quad \Delta \theta_i \quad \Delta u_j \quad \Delta v_j \quad \Delta \theta_j\}^T$$

$$[K_L] = \begin{bmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 & 6EI/\ell^2 & 0 & -12EI/\ell^3 & 6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 4EI/\ell & 0 & -6EI/\ell^2 & 2EI/\ell \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 & -6EI/\ell^2 & 0 & 12EI/\ell^3 & -6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 2EI/\ell & 0 & -6EI/\ell^2 & 4EI/\ell \end{bmatrix}$$

$$[K_G] = P \cdot \begin{bmatrix} 1/\ell & 0 & 0 & -1/\ell & 0 & 0 \\ 0 & 6/5\ell & 1/10 & 0 & -6/5\ell & 1/10 \\ 0 & 1/10 & 2\ell/15 & 0 & -1/10 & -\ell/30 \\ -1/\ell & 0 & 0 & 1/\ell & 0 & 0 \\ 0 & -6/5\ell & -1/10 & 0 & 6/5\ell & -1/10 \\ 0 & 1/10 & -\ell/30 & 0 & -1/10 & 2\ell/15 \end{bmatrix}$$

- $[K_T]$  : Tangential stiffness matrix of an element  
 $[K_L]$  : Linear stiffness matrix of an element  
 $[K_G]$  : Geometrical non-linear stiffness matrix of an element  
 $P$  : Axial force of an element (positive sign means tension)  
 $\{\Delta f\}$  : Incremental nodal force vector of an element  
 $\{\Delta u\}$  : Incremental nodal displacement vector of an element  
 $E, A, I, \ell$  : Elastic modulus, section area, moment of inertia, element length  
 $\Delta N, \Delta S, \Delta M$  : Increments of axial force, shear force and bending moment  
 $\Delta u, \Delta v, \Delta \theta$  : Increments of displacements in axial direction, transverse direction and rotation  
 Subscripts  $i$  and  $j$  : variables for node  $i$  and node  $j$

## 11.2 Calculation formulas for Internal forces

### 11.2.1 Calculation of internal force

The increments of internal force  $\{\Delta f^*\}$  can be obtained using  $[K_L]$  and  $\{\Delta u^*\}$ . Where,  $[K_L]$  is a linear stiffness matrix of an element, and  $\{\Delta u^*\}$  is an incremental displacement vector excluding rigid rotation in local coordinate system.

$$\{\Delta f^*\} = [K_L]\{\Delta u^*\}$$

$$\{\Delta f^*\} = \{\Delta N_i^* \quad \Delta S_i^* \quad \Delta M_i^* \quad \Delta N_j^* \quad \Delta S_j^* \quad \Delta M_j^*\}^T$$

$$\{\Delta u^*\} = \{\Delta u_i^* \quad \Delta v_i^* \quad \Delta \theta_i^* \quad \Delta u_j^* \quad \Delta v_j^* \quad \Delta \theta_j^*\}^T$$

$$\Delta u_i^* = 0$$

$$\Delta v_i^* = 0$$

$$\Delta \theta_i^* = (\tan \theta_i^*)_k - (\tan \theta_i^*)_{k-1}$$

$$\Delta u_j^* = \Delta \ell$$

$$\Delta v_j^* = 0$$

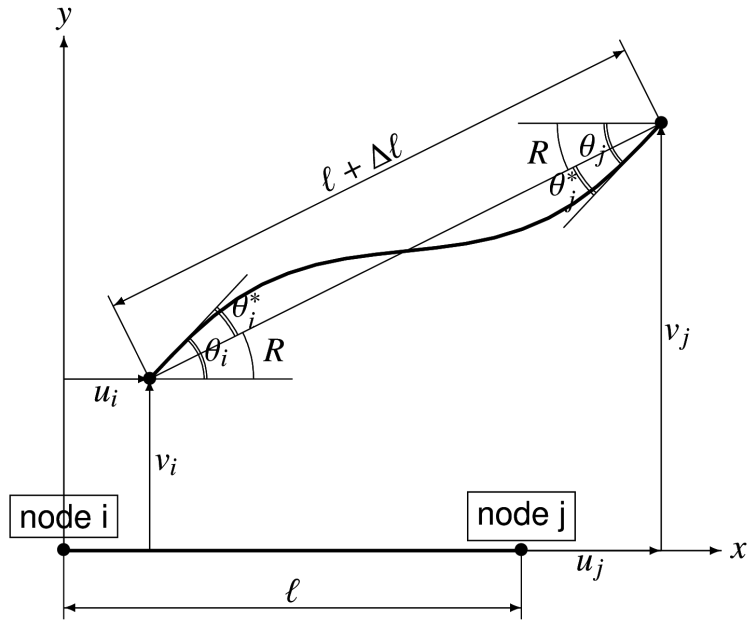
$$\Delta \theta_j^* = (\tan \theta_j^*)_k - (\tan \theta_j^*)_{k-1}$$

Where,  $\Delta \ell$  means the difference of the previous element length and current element length. Regarding the rotation component, it is taken as the difference of the previous rotation angle (subscript  $k-1$ ) and current rotation angle (subscript  $k$ ).

### 11.2.2 Elimination of rigid rotation

The method of elimination of rigid rotation is shown below: Using the addition formula for tangent,

$$\begin{aligned} \frac{dv^*}{dx^*}|_i &= \tan \theta_i^* = \tan(\theta_i - R) = \frac{\tan \theta_i - \tan R}{1 + \tan \theta_i \tan R} = \frac{(\ell + u_j - u_i) \tan \theta_i - (v_j - v_i)}{(\ell + u_j - u_i) + (v_j - v_i) \tan \theta_i} \\ \frac{dv^*}{dx^*}|_j &= \tan \theta_j^* = \tan(\theta_j - R) = \frac{\tan \theta_j - \tan R}{1 + \tan \theta_j \tan R} = \frac{(\ell + u_j - u_i) \tan \theta_j - (v_j - v_i)}{(\ell + u_j - u_i) + (v_j - v_i) \tan \theta_j} \\ \left( \tan R &= \frac{v_j - v_i}{\ell + u_j - u_i}, \quad \frac{dv}{dx}|_i = \tan \theta_i, \quad \frac{dv}{dx}|_j = \tan \theta_j \right) \end{aligned}$$



Using above equations,  $(\tan \theta_i^*)_k, (\tan \theta_j^*)_k$  for current step and  $(\tan \theta_i^*)_{k-1}, (\tan \theta_j^*)_{k-1}$  for previous step can be obtained. From these, rotation angles excluding rigid rotation in an element can be obtained as follows.

$$\Delta \theta_i^* = (\tan \theta_i^*)_k - (\tan \theta_i^*)_{k-1}$$

$$\Delta \theta_j^* = (\tan \theta_j^*)_k - (\tan \theta_j^*)_{k-1}$$

In these steps, transformation matrix which is calculated from original (initial) coordinates of nodes should be used.

### 11.3 Arc-Length Method

When simplified scalar load-displacement curve is considered, following equations can be obtained referring below figure.

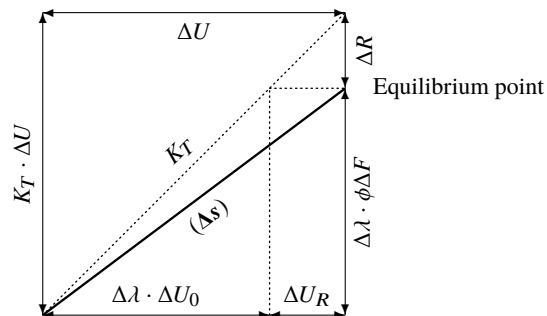
$$K_T \cdot \Delta U = \Delta \lambda \cdot \phi \Delta F + \Delta R$$

$$\Delta U = \Delta \lambda \cdot \Delta U_0 + \Delta U_R$$

$$(\Delta s)^2 = (\Delta U)^2 + (\Delta \lambda \cdot \phi \Delta F)^2$$

$$= (\Delta \lambda)^2 \{ (\Delta U_0)^2 + (\phi \Delta F)^2 \} + 2 \lambda \cdot (\Delta U_0 \cdot \Delta U_R) + (\Delta U_R)^2$$

$$\Delta \lambda = \frac{-(\Delta U_0 \cdot \Delta U_R) \pm \sqrt{\{ (\Delta U_0)^2 + (\phi \Delta F)^2 \} \cdot (\Delta s)^2 - (\phi \Delta F \cdot \Delta U_R)^2}}{(\Delta U_0)^2 + (\phi \Delta F)^2}$$





where, $K_T$	: Tangential stiffness	$\Delta\lambda$	: Coefficient for external force
$\Delta F$	: External force increment	$\Delta U_0$	: Displacement increment for external force
$\Delta R$	: Unbalanced force increment	$\Delta U_R$	: Displacement increment for unbalanced
$\Delta U$	: Displacement increment	$\Delta s$	: Arc length
		$\phi$	: Scaling parameter

### 11.3.1 Initial value of $\Delta\lambda$

If  $\Delta U_R = 0$  is assumed, initial value of  $\Delta\lambda$  can be obtained as following equation.

$$\Delta\lambda_0 = \pm \sqrt{\frac{(\Delta s)^2}{(\Delta U_0)^2 + (\phi\Delta F)^2}}$$

Above equation can take two values, and it should be noted that a sign of  $\Delta\lambda_0$  is very important in Arc-Length method. A sign of  $\Delta\lambda_0$  is defined as shown below:

- Define a displacement increment vector  $\{\Delta U_{-1}\}$  from previous equilibrium point to current equilibrium point.
- Calculate  $\{\Delta U_0\}$  as  $\{\Delta U_0\} = [K_T]^{-1}\{\Delta F\}$ .
- Calculate an inner product  $\{\Delta U_{-1}\}^T \{\Delta U_0\} = |\Delta U_{-1}| \cdot |\Delta U_0| \cdot \cos \theta$ , where  $\theta$  is an angle between 2 displacement increment vectors.
- If an inner product  $\{\Delta U_{-1}\}^T \{\Delta U_0\} \geq 0$ , the angle  $\theta$  is less than or equal to 90 degree. In this case,  $\Delta\lambda_0$  has positive sign.
- If an inner product  $\{\Delta U_{-1}\}^T \{\Delta U_0\} < 0$ , the angle  $\theta$  is greater than 90 degree. In this case,  $\Delta\lambda_0$  has negative sign.

Regarding the scaling parameter  $\phi$ , it can be obtained as following equation. In this program, recommended value of  $\alpha$  is one ( $\alpha = 1.0$ ).

$$\phi = \sqrt{\frac{\alpha}{\{\Delta F\}^T \{\Delta F\}}}$$

Regarding the arc length  $\Delta s$ , it can be obtained assuming  $\Delta\lambda = 1.0$ .

$$\Delta s = \sqrt{\{\Delta U_0\}^T \{\Delta U_0\} + \phi^2 \cdot \{\Delta F\}^T \{\Delta F\}} \quad (\Delta\lambda_0 = 1.0)$$

### 11.3.2 Correction factor $\Delta\lambda$ for iterative calculation

Referring above conceptual figure and replacing  $\Delta s$  to  $\Delta L$ , following equation can be obtained.

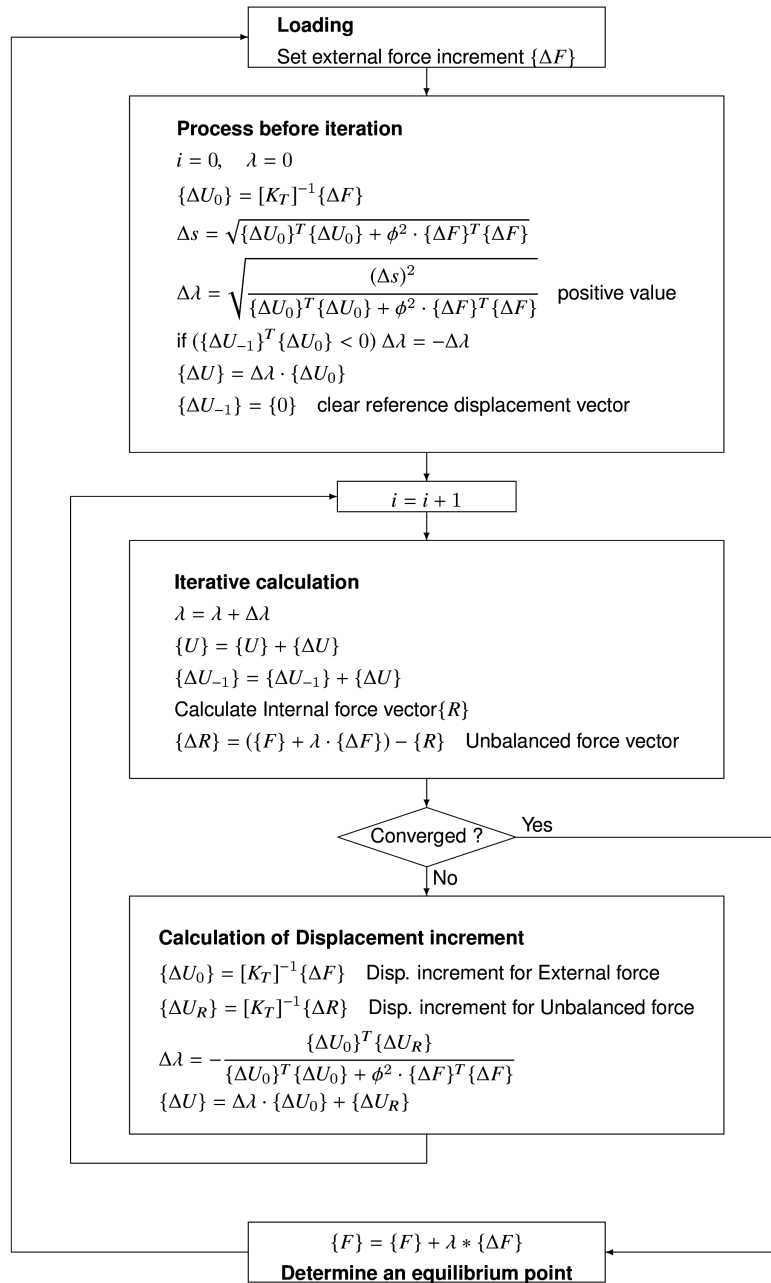
$$\begin{aligned} (\Delta L)^2 &= (\Delta U)^2 + (\Delta\lambda \cdot \phi\Delta F)^2 \\ &= (\Delta\lambda)^2 \{(\Delta U_0)^2 + (\phi\Delta F)^2\} + 2\Delta\lambda \cdot (\Delta U_0 \cdot \Delta U_R) + (\Delta U_R)^2 \end{aligned}$$

From the condition of minimization of  $(\Delta L)^2$ ,  $\Delta\lambda$  can be calculate as following equation.

$$\frac{d(\Delta L)^2}{d\Delta\lambda} = 0 \quad \rightarrow \quad \Delta\lambda = -\frac{\Delta U_0 \cdot \Delta U_R}{(\Delta U_0)^2 + (\phi\Delta F)^2}$$

### 11.3.3 Flowchart for analysis

$\{F\}$	: Total external force vector at equilibrium point
$\{\Delta F\}$	: External incremental force vector
$\{R\}$	: Internal force vector
$\{\Delta R\}$	: Unbalanced force vector
$\{U\}$	: Total displacement vector
$\{\Delta U\}$	: Displacement increment vector
$\{\Delta U_0\}$	: Displacement increment vector for external force increment
$\{\Delta U_R\}$	: Displacement increment vector for unbalanced force
$\{\Delta U_{-1}\}$	: Displacement increment vector from previous equilibrium point to current equilibrium point
$[K_T]$	: Tangential stiffness matrix including non-linear component
$\Delta s$	: Arc length
$\lambda$	: Coefficient for external force increment and displacement increment
$\Delta\lambda$	: Increment of coefficient $\lambda$
$\phi$	: Scaling parameter



Note) Condition for convergence:  $|\Delta U_i / U_i| < 1e-3$

A treatment of  $\Delta s$  and  $\Delta \lambda$  in actual programming code is shown below:

- At the initial loading (nnn=1),  $\Delta s$  is calculated and  $\Delta \lambda_0$  is set to 1.0.
- If calculated  $\Delta \lambda$  is much less than  $\Delta \lambda_0$ , the value of  $\Delta s$  should be increased, because  $\Delta s$  has been calculated with the assumption of  $\Delta \lambda = 1.0$ .
- If calculated  $\Delta \lambda$  is greater than initial value of  $\Delta \lambda_0 (= 1.0)$ , the increase of the value of  $\Delta s$  should be stopped, because huge value of  $\Delta s$  causes unexpected behavior of the structure.

```

# Initial parameter setting for Arc-length method
if nnn==1:
    ds0=np.sqrt(np.sum(dis0*dis0)+spara*spara*np.sum(df*df))
    ds=ds0
    dlam0=1.0
dlam=np.sqrt(ds*ds/(np.sum(dis0*dis0)+spara*spara*np.sum(df*df)))
if np.abs(dlam) < 0.1*np.abs(dlam0): ds=ds*1.2
if np.abs(dlam0) < np.abs(dlam): ds=ds
dlam=np.sqrt(ds*ds/(np.sum(dis0*dis0)+spara*spara*np.sum(df*df)))
if np.sum(dis_ref*dis0)<0.0: dlam=-dlam
  
```

## 12 2D Frame Buckling Analysis

### 12.1 Characteristic equation for buckling analysis

Characteristic equation to obtain the buckling load and displacement mode of 2D frame structure is shown below. Although the shapes of the matrices  $[K_L]$  and  $[K_G]$  are same as those of 2D frame geometrically nonlinear analysis, the compressive axial force  $P$  included in  $[K_G]$  shall have the positive sign. The coefficient  $\lambda$  can be obtained by solving the following characteristic equation. However, since  $[K_G]$  includes the axial force  $P$ , it is necessary to carry out the one time linear analysis to obtain the axial force  $P$  before eigenvalue analysis.

$$\{[K_L] - \lambda[K_G]\} \{u\} = \{0\}$$

$$[K_L] = \begin{bmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 & 6EI/\ell^2 & 0 & -12EI/\ell^3 & 6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 4EI/\ell & 0 & -6EI/\ell^2 & 2EI/\ell \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 & -6EI/\ell^2 & 0 & 12EI/\ell^3 & -6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 2EI/\ell & 0 & -6EI/\ell^2 & 4EI/\ell \end{bmatrix}$$

$$[K_G] = P \cdot \begin{bmatrix} 1/\ell & 0 & 0 & -1/\ell & 0 & 0 \\ 0 & 6/5\ell & 1/10 & 0 & -6/5\ell & 1/10 \\ 0 & 1/10 & 2\ell/15 & 0 & -1/10 & -\ell/30 \\ -1/\ell & 0 & 0 & 1/\ell & 0 & 0 \\ 0 & -6/5\ell & -1/10 & 0 & 6/5\ell & -1/10 \\ 0 & 1/10 & -\ell/30 & 0 & -1/10 & 2\ell/15 \end{bmatrix}$$

- $[K_L]$  : element linear stiffness matrix  
 $[K_G]$  : geometrically nonlinear term of element stiffness matrix  
 $P$  : element axial force (compression is positive)  
 $\{u\}$  : displacement of element  
 $E, A, I, \ell$  : elastic modulus, section area, moment of inertia, element length  
 subscript  $i, j$  : node- $i$  and node- $j$

## 13 1D Thermal Conductivity Analysis

### 13.1 Solution of unsteady finite element equation

Equation to be solved by Crank-Nicolson method is shown below.

$$\left(\frac{1}{2}[\mathbf{K}] + \frac{1}{\Delta t}[\mathbf{C}]\right) \{\Phi(t + \Delta t)\} = \left(-\frac{1}{2}[\mathbf{K}] + \frac{1}{\Delta t}[\mathbf{C}]\right) \{\Phi(t)\} + \frac{\{\mathbf{F}(t + \Delta t)\} + \{\mathbf{F}(t)\}}{2}$$

- $[\mathbf{K}]$  : heat conduction matrix       $[\mathbf{C}]$  : heat capacity matrix  
 $\{\Phi\}$  : nodal temperature vector       $\{\mathbf{F}\}$  : heat flux vector

Each matrix and vector for an element can be expressed as follows.

$$[\mathbf{k}] = \frac{\kappa \cdot A}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \alpha_{ci} \cdot A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_{cj} \cdot A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\mathbf{c}] = \rho \cdot c \cdot \ell \cdot A \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

$$\{\mathbf{f}\} = \frac{\dot{Q} \cdot \ell \cdot A}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \alpha_{ci} \cdot T_{ci} \cdot A \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \alpha_{cj} \cdot T_{cj} \cdot A \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

- $\kappa$  : element heat conductivity coefficient       $c$  : element specific heat       $\rho$  : element density  
 $\ell$  : element length       $A$  : element section area       $\dot{Q}$  : element heat rate

- $\alpha_{ci}$  : heat transfer rate of node- $i$  (if no-heat transfer boundary:  $\alpha_{ci} = 0$ )  
 $\alpha_{cj}$  : heat transfer rate of node- $j$  (if no-heat transfer boundary:  $\alpha_{cj} = 0$ )  
 $T_i$  : outside temperature of heat transfer boundary at node- $i$   
 $T_j$  : outside temperature of heat transfer boundary at node- $j$

### 13.1.1 Explanation of heating material

Treatment of Heating material such as cement concrete is considered. The adiabatic temperature rise is assumed as follow.

$$T = K \cdot (1 - e^{-\alpha \cdot t})$$

$T$ : adiabatic temperature rise                       $K$ : maximum temperature rise  
 $\alpha$ : parameter for heat generation rate       $t$ : time

$K$  (Tk) and  $\alpha$  (Al) in above equation will be inputted as material characteristics. Using above, heat value  $Q$  and heat rate  $\dot{Q}$  can be expressed as follow.

$$Q = \rho \cdot c \cdot T(t) \quad \rightarrow \quad \dot{Q} = \rho \cdot c \cdot T_k \cdot \alpha \cdot e^{-\alpha t}$$

## Appendix A Treatment of known and unknown variables in simultaneous linear equations

In the FEM programs introduced in the site, simultaneous linear equations are solved without reducing the number of variables after introducing boundary conditions in the equations. In this case, swapping of known variables and unknown variables is required in the simultaneous linear equations. The swapping method of variables is introduced in this page.

As a simple example, following simultaneous equations with 3 variables are considered.

$$\begin{aligned} k_{11}x_1 + k_{12}x_2 + k_{13}x_3 &= f_1 \\ k_{21}x_1 + k_{22}x_2 + k_{23}x_3 &= f_2 \\ k_{31}x_1 + k_{32}x_2 + k_{33}x_3 &= f_3 \end{aligned}$$

It is necessary to rearrange the equations shown below to carry out the matrix operation, where  $x_1, x_3, f_2$  are unknown variables,  $f_1, f_3, x_2$  are known variables.

$$\begin{aligned} k_{11}x_1 + 0 + k_{13}x_3 &= f_1 - k_{12}x_2 \\ k_{21}x_1 - f_2 + k_{23}x_3 &= 0 - k_{22}x_2 \\ k_{31}x_1 + 0 + k_{33}x_3 &= f_3 - k_{32}x_2 \end{aligned}$$

Considering above, following general expression can be obtained.

### A.0.1 Original stiffness equation

$$\begin{bmatrix} k_{11} & \dots & k_{1i} & \dots & k_{1j} & \dots & k_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{i1} & \dots & k_{ii} & \dots & k_{ij} & \dots & k_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{j1} & \dots & k_{ji} & \dots & k_{jj} & \dots & k_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{n1} & \dots & k_{ni} & \dots & k_{nj} & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \vdots \\ \delta_i \\ \vdots \\ \delta_j \\ \vdots \\ \delta_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_j \\ \vdots \\ f_n \end{Bmatrix}$$

### A.0.2 Stiffness equation after introducing the boundary conditions

The treatment that the locations of  $k_{ii}$  and  $k_{jj}$  are set to 1 and other elements in the column  $i$  and column  $j$  are set to zero shall be done under the condition of known variables of  $\delta_i$  and  $\delta_j$ . And the effects related to column  $i$  and column  $j$  shall be transposed to right side.

$$\begin{bmatrix} k_{11} & \dots & 0 & \dots & 0 & \dots & k_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{i1} & \dots & 1 & \dots & 0 & \dots & k_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{j1} & \dots & 0 & \dots & 1 & \dots & k_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{n1} & \dots & 0 & \dots & 0 & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \vdots \\ -f_i \\ \vdots \\ -f_j \\ \vdots \\ \delta_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ f_n \end{Bmatrix} - \delta_i \begin{Bmatrix} k_{1i} \\ \vdots \\ k_{ii} \\ \vdots \\ k_{ji} \\ \vdots \\ k_{ni} \end{Bmatrix} - \delta_j \begin{Bmatrix} k_{1j} \\ \vdots \\ k_{ij} \\ \vdots \\ k_{jj} \\ \vdots \\ k_{nj} \end{Bmatrix}$$

Although the stiffness matrix becomes asymmetry by this treatment, the simultaneous equations with thousands variables can be solved using `numpy.linalg ( x = np.linalg.solve(A, b )`) without any stress, because the numpy function has good performance.

### A.0.3 Actural treatment in the program

Some parts of actural Python code for structural analysis are shown below. It shall be noted that known variables shall be set again into the answer displacements after solving the simultaneous equations.

```

# treatment of boundary conditions
for i in range(0,npoin):
    for j in range(0,nfree):
        if mpfix[j,i]==1:
            iz=i*nfree+j
            fp[iz]=0.0
            for k in range(0,n):
                fp[k]=fp[k]-rdis[j,i]*gk[k,iz]
                gk[k,iz]=0.0
            gk[iz,iz]=1.0

# solution of simultaneous linear equations
disg = np.linalg.solve(gk, fp)

# recovery of restricted displacements
for i in range(0,npoin):
    for j in range(0,nfree):
        if mpfix[j,i]==1:
            iz=i*nfree+j
            disg[iz]=rdis[j,i]

```

npoin : number of nodes  
 nfree : degree of freedom of a node  
 mpfix : array for boundary conditions for all nodes  
 fp : nodal external force vector  
 rdis : forced displacement at specified nodes  
 gk : stiffness matrix in global coordinate  
 disg : solution (displacement) of stiffness equation

## Appendix B Preamble and structure of this document

```

\documentclass[english,10pt]{jsarticle}

\usepackage[T1]{fontenc}
\usepackage{newtxtext,newtxmath}

\usepackage{bm}
\usepackage{epic,eepic,eepicsup}
\usepackage{float} %\begin{figure}[H]: fix the location of figure
\usepackage{color,fancybox}
\usepackage{ascmac}
\usepackage{listings}
\usepackage[a4paper,top=20mm,bottom=20mm,left=20mm,right=20mm]{geometry}
\usepackage[dvipdfmx]{graphicx}
\setcounter{tocdepth}{3}
\renewcommand{\appendixname}{Appendix~}

\begin{document}
\pagenumbering{roman}
\setcounter{page}{1}
\rightline{\today}
\vspace{1zh}
\centerline{\Large\bfseries Description of FEM theory for programing}
\tableofcontents
\clearpage

\pagenumbering{arabic}
\setcounter{page}{1}

\section{xxxxx}

\section{xxxxx}
.....

\appendix
\section{xxxxx}
.....
\end{document}

```