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Nemerical Integration of Equation of Motion for SDOF system using Runge-Kutta method

1. Equation of motion

The equation of motion for Single Degree of Freedom system (SDOF) can be expressed as follow:

$$m \cdot \ddot{x} + c \cdot \dot{x} + k \cdot x = f(t) \tag{1}$$

where,	m	: mass	$\ddot{x}$	: acceleration	f	: external force
	c	: damping	$\dot{x}$	: velocity		
	k	: stiffness	x	: displacement		

2. Runge-Kutta method

In order to apply Runge-Kutta method, it is necessary to convert the equation of motion to simultaneous differential equations shown below.

$$\begin{cases} f_a(t,x,y) = \dot{x} = \dfrac{dx}{dt} = y \\[2ex] f_b(t,x,y) = \ddot{x} = \dfrac{dy}{dt} = \dfrac{f(t)}{m} - \dfrac{c}{m} \cdot y - \dfrac{k}{m} \cdot x \end{cases}$$

The simultaneous differential equations shown above can be solved applying the Runge-Kutta method. The procedure for Runge-Kutta method is shown below, where h is a time increment of calculation. Please take care to provide initial values for each variable of t, x and y.

$$\begin{cases} k_{a1} = f_a(t_n, x_n, y_n) \\ k_{a2} = f_a(t_n + h/2, x_n + h/2 \cdot k_{a1}, y_n + h/2 \cdot k_{a1}) \\ k_{a3} = f_a(t_n + h/2, x_n + h/2 \cdot k_{a2}, y_n + h/2 \cdot k_{a2}) \\ k_{a4} = f_a(t_n + h, x_n + h \cdot k_{a3}, y_n + h \cdot k_{a3}) \\ x_{n+1} = x_n + h/6 \cdot (k_{a1} + 2k_{a2} + 2k_{a3} + k_{a4}) \end{cases}$$

$$\begin{cases} k_{b1} = f_b(t_n, x_n, y_n) \\ k_{b2} = f_b(t_n + h/2, x_n + h/2 \cdot k_{b1}, y_n + h/2 \cdot k_{b1}) \\ k_{b3} = f_b(t_n + h/2, x_n + h/2 \cdot k_{b2}, y_n + h/2 \cdot k_{b2}) \\ k_{b4} = f_b(t_n + h, x_n + h \cdot k_{b3}, y_n + h \cdot k_{b3}) \\ y_{n+1} = y_n + h/6 \cdot (k_{b1} + 2k_{b2} + 2k_{b3} + k_{b4}) \end{cases}$$

$$\begin{cases} \ddot{x} = \dfrac{f(t)}{m} - \dfrac{c}{m} \cdot y_{n+1} - \dfrac{k}{m} \cdot x_{n+1} & \text{(acceleration)} \\ \dot{x} = y_{n+1} & \text{(velocity)} \\ x = x_{n+1} & \text{(displacement)} \end{cases}$$

3. Stability of Runge-Kutta method

We study the stability condition of Runge-Kutta method on the numerical integration of equation of motion. For this purpose, we consider the simplest equation shown below.

$$\dot{z} = i \cdot \omega \cdot z \tag{2}$$

Abeve equation is a differential equation related the velocity ($\dot{z}$) and the displacement (z) expressed in complex number, and ω means natural angular frequency.

Adopt Runge-Kutta method to above differential iquation.

$$\begin{aligned}
k_1 =& f(z_n) && = i\omega z \\
k_2 =& f(z_n + h/2 \cdot k_1) && = (i\omega - h/2 \cdot \omega^2) z_n \\
k_3 =& f(z_n + h/2 \cdot k_2) && = (i\omega - h/2 \cdot \omega^2 - i \cdot h^2/4 \cdot \omega^3) z_n \\
k_4 =& f(z_n + h \cdot k_2) && = (i\omega - h \cdot \omega^2 - i \cdot h^2/2 \cdot \omega^3 + h^3/4 \cdot \omega^4) z_n
\end{aligned}$$

$$\begin{aligned}
z_{n+1} =& z_n + h/6(k_1 + 2k_2 + 2k_3 + k_4) \\
=& z_n \left\{ 1 - \frac{1}{2}h^2\omega^2 + \frac{1}{24}h^4\omega^4 + i\left(h\omega - \frac{1}{6}h^3\omega^3 \right) \right\} \\
=& z_n \left\{ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + i\left(x - \frac{1}{6}x^3 \right) \right\} \qquad \text{where, } x = h\omega.
\end{aligned}$$

From above, following relationship can be obtained.

$$\frac{z_{n+1}}{z_n} = \left\{ 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + i\left(x - \frac{1}{6}x^3 \right) \right\} = A \cdot e^{i\theta} \qquad \text{where, } x = h\omega.$$

The stability condition of this numerical integration is that 'A shall be less than or equal to 1.' This condition can be expressed below.

$$A^2 = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \right)^2 + \left(x - \frac{1}{6}x^3 \right)^2 = 1 + \frac{x^6}{576}(x^2 - 8) \leqq 1$$

$$x = h\omega \leqq 2\sqrt{2} \quad \Rightarrow \quad h \leqq \frac{2\sqrt{2}}{2\pi}T \risingdotseq 0.450T \qquad \text{where, } T \text{ is a natural period}$$

From above, if we want to know the behavior of the structure which has the natural period of 0.02 second, the time increment for Runge-Kutta method shall be less than 0.009 second (0.02× 0.45).

4. Calculation example

As a example of use of numerical integration of equation of motion, the calculation of the responce spectrum for the earthquake wave acceleration was done using Runge-Kutta method.

Conditions for calculation of SDOF system is shown below.

$$m = 1 \qquad k = \frac{4\pi^2 m}{T^2} \qquad c = 2h\sqrt{km}$$

Time interval of the ground acceleration record	(sec)	$\Delta t = 0.01$
Considered range of natural period of SDOF system	(sec)	$T = 0.02 \sim 10$
Damping factor		$h = 0.05$

In addition, following were considered for calculation.

- ◯ The interval of data output is the same as data input interval (0.01sec).
- ◯ The interval of calculation by Runge-Kutta method is reduced to 1/5 (0.002sec) of the data input interval. The interval of calculation shall be changed taking into consuderation of the stability of numerical integration.
- ◯ Because, if the same time interval (0.01sec) as input data is used for calculation, the condition of stability can not be satisfied.
- ◯ The ground acceleration value is linearly interpolated for small time interval for calculation.

The results of calculation are shown below. The result by Dr.Ohsaki's program is drawn as red line, and result by Runge-Kutta method is drawn as black line. The degree of coincidence with red line and black line, the result expected is obtained.

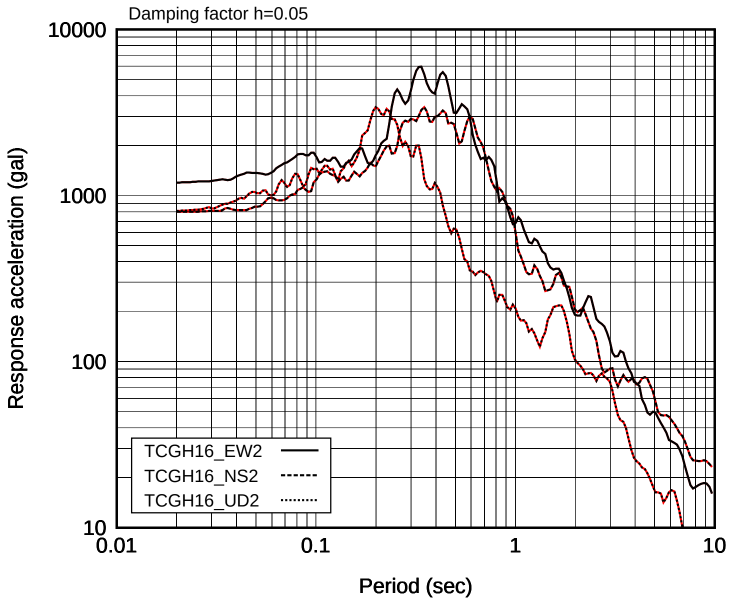

Fig.1 Response acceleration spectrum

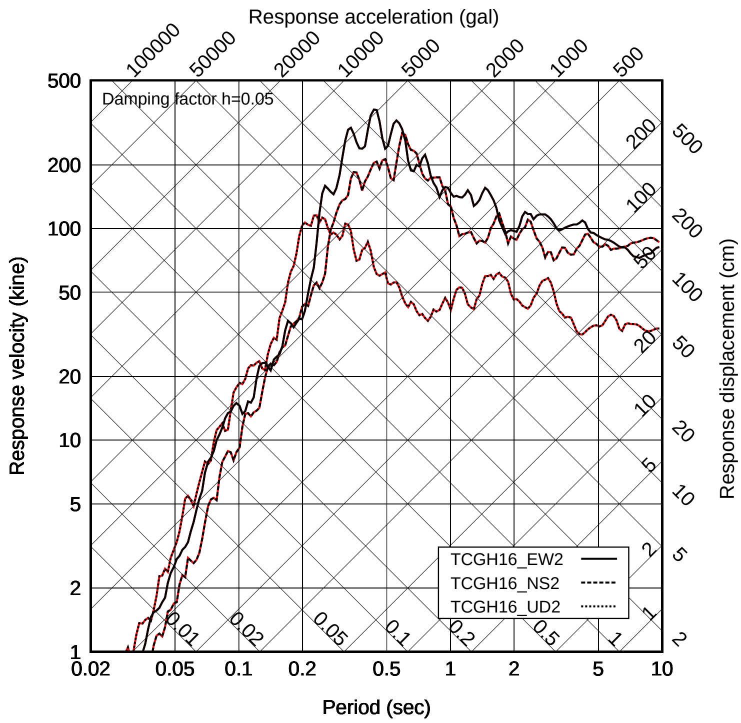

Fig.2 Response velocity spectrum