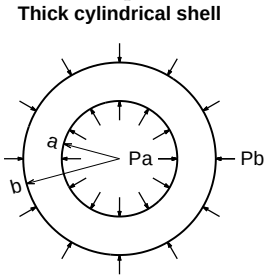
November 18, 2012

Example of Axisymmetric Stress Analysis

1. Comparison with theoretical solution

Consider the circular cylindrical shell with thick wall under uniform internal pressure and uniform external pressure in plane strain state. Material has the properties of elastic modulus $E = 25,000N/mm^2$, Poisson' ratio $\nu = 0.2$. σ_r, σ_θ and u mean normal stress in radius direction, normal stress in circumference direction and displacement in radius direction for each.



$$
\begin{cases}
u = \dfrac{a^2}{b^2-a^2}\left(\dfrac{(1+\nu)(1-2\nu)}{E}\cdot r + \dfrac{1+\nu}{E}\cdot\dfrac{b^2}{r}\right)\cdot P_a \\
\qquad - \dfrac{b^2}{b^2-a^2}\left(\dfrac{(1+\nu)(1-2\nu)}{E}\cdot r + \dfrac{1+\nu}{E}\cdot\dfrac{a^2}{r}\right)\cdot P_b \\
\sigma_r = \dfrac{a^2}{b^2-a^2}\left(1-\dfrac{b^2}{r^2}\right)\cdot P_a - \dfrac{b^2}{b^2-a^2}\left(1-\dfrac{a^2}{r^2}\right)\cdot P_b \\
\sigma_\theta = \dfrac{a^2}{b^2-a^2}\left(1+\dfrac{b^2}{r^2}\right)\cdot P_a - \dfrac{b^2}{b^2-a^2}\left(1+\dfrac{a^2}{r^2}\right)\cdot P_b
\end{cases}
$$

① Axisymmetric FEM (f90_FEM_ASNT.exe, 12 nodes, 5 elements)

a	b	P_a	P_b	$\sigma_{r(a)}$	$\sigma_{\theta(a)}$	$\sigma_{r(b)}$	$\sigma_{\theta(b)}$	u_a	u_b
3000	3600	1	0	-0.873	5.420	-0.078	4.624	0.667	0.628
4000	4800	1	0	-0.873	5.420	-0.078	4.624	0.890	0.838
5000	6000	1	0	-0.873	5.420	-0.078	4.624	1.112	1.047
3000	3600	0	1	-0.127	-6.420	-0.922	-5.624	-0.754	-0.732
4000	4800	0	1	-0.127	-6.420	-0.922	-5.624	-1.005	-0.976
5000	6000	0	1	-0.127	-6.420	-0.922	-5.624	-1.256	-1.220

② Theoretical solution

a	b	P_a	P_b	$\sigma_{r(a)}$	$\sigma_{\theta(a)}$	$\sigma_{r(b)}$	$\sigma_{\theta(b)}$	u_a	u_b
3000	3600	1	0	-1.000	5.545	0.000	4.545	0.668	0.628
4000	4800	1	0	-1.000	5.545	0.000	4.545	0.890	0.838
5000	6000	1	0	-1.000	5.545	0.000	4.545	1.113	1.047
3000	3600	0	1	0.000	-6.545	-1.000	-5.545	-0.754	-0.732
4000	4800	0	1	0.000	-6.545	-1.000	-5.545	-1.005	-0.976
5000	6000	0	1	0.000	-6.545	-1.000	-5.545	-1.257	-1.220

Ratio ①/②

a	b	P_a	P_b	$\sigma_{r(a)}$	$\sigma_{\theta(a)}$	$\sigma_{r(b)}$	$\sigma_{\theta(b)}$	u_a	u_b
3000	3600	1	0	0.873	0.977	–	1.017	0.999	1.000
4000	4800	1	0	0.873	0.977	–	1.017	1.000	1.000
5000	6000	1	0	0.873	0.977	–	1.017	0.999	1.000
3000	3600	0	1	–	0.981	0.922	1.014	1.000	1.000
4000	4800	0	1	–	0.981	0.922	1.014	1.000	1.000
5000	6000	0	1	–	0.981	0.922	1.014	0.999	1.000

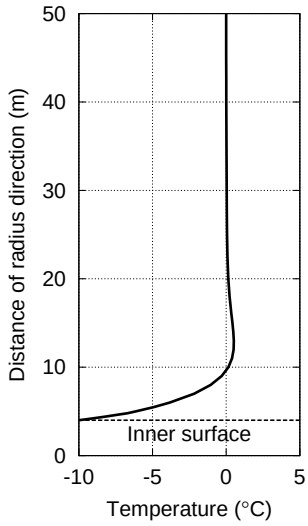
- ◯ In axisymmetric FEM model, number of division in radius direction is 5, and number of division is 1. Displacements in axial direction are restricted in order to satisfy the condition of plane strain state.
- ◯ In theoretical solution, normal stresses at inner surface and outer surface of wall are obtained under strict boundary conditions. The other hand, in axisymmetric FEM, normal stresses shown above tables are mean values of inner side element or outer side element. So, the difference is found between theoretical solution and results of axisymmetric FEM. But the difference is not so large.
- ◯ In displacement, the difference between theoretical solution and axisymmetric FEM result is small and it has the accuracy of 0.1%.

2. Comparison with 2D FEM

2.1 Model

- ◯ By axisymmetric FEM and 2D FEM, the comparison of results were performed.
- ◯ Characteristics of model for analysis is shown in following table.
- ◯ In axisymmetric FEM model, number of division in radius direction is the same value as 2D FEM. (number of division is 33)
- ◯ Stress state was plane strain.
- ◯ The comparable result was obtained by axisymmetric FEM and 2D FEM.

Section	Double reinforced
Internal diameter of tunnel	4,000 mm
Thickness of concrete wall	800 mm
Outer diameter of domain	50,000 mm
Cover for Re-bar	100 mm
Inner re-bar (equivalent steel thickness)	D32@200x2 3.97 mm
Outer re-bar (equivalent steel thickness)	D32@200x2 3.97 mm
Elastic modulus of concrete Poisson's ratio of concrete Thermal expansion coefficient of concrete	25,000 N/mm^2 0.2 (zero at No-tension state) 10×10^{-5} °C^{-1}
Elastic modulus of re-bar Poisson's ratio of re-bar Thermal expansion coefficient of re-bar	200,000 N/mm^2 0.3 10×10^{-6} °C^{-1}
Elastic modulus of bed rock Poisson's ratio of bed rock Thermal expansion coefficient of bed rock	1∼100,000 N/mm^2 0.25 7×10^{-6} °C^{-1}
Internal water pressure	1 MPa
Temperature change in the domain (by heat conduction analysis)	(see right figure) (−10 °C at inner surface)



(1) Results

① Axisymmetric FEM(f90_FEM_ASNT.exe, 68 nodes, 33 elements)

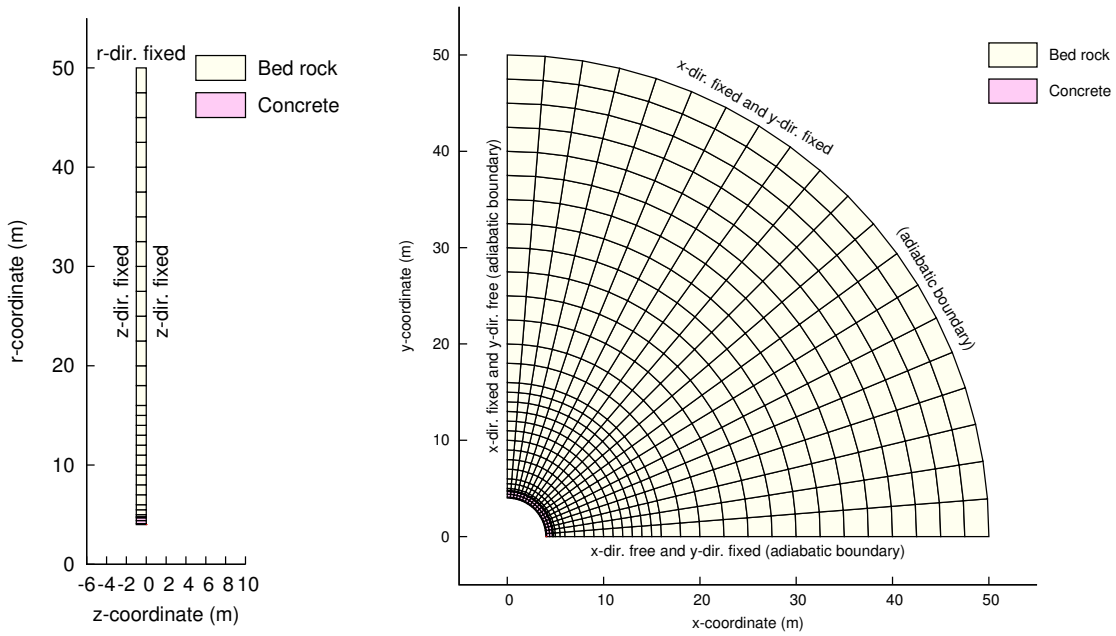
E_g	σ_{cr}	σ_{sa}	σ_{sb}	$\sigma_{\theta g}$	σ_{rg}	u_a	u_b
1	-0.987	540.244	465.326	0.002	-0.002	9.596	9.508
10	-0.987	530.949	457.201	0.015	-0.015	9.423	9.335
100	-0.987	453.784	389.749	0.131	-0.130	7.983	7.893
1000	-0.987	197.260	165.513	0.550	-0.511	3.195	3.102
10000	-0.987	54.594	40.804	1.278	-0.712	0.532	0.438
100000	-0.987	33.539	22.399	6.558	-0.630	0.139	0.044

② 2D plane strain FEM(f90_FEM_PLNT.exe, 714 nodes, 660 elements)

E_g	σ_{cr}	σ_{sa}	σ_{sb}	$\sigma_{\theta g}$	σ_{rg}	u_a	u_b
1	-0.987	539.839	465.177	0.002	-0.002	9.590	9.502
10	-0.987	530.549	457.058	0.015	-0.015	9.417	9.329
100	-0.987	453.437	389.645	0.131	-0.130	7.978	7.889
1000	-0.987	197.121	165.487	0.549	-0.511	3.193	3.100
10000	-0.987	54.576	40.806	1.278	-0.711	0.532	0.437
100000	-0.987	33.537	22.404	6.558	-0.628	0.139	0.044

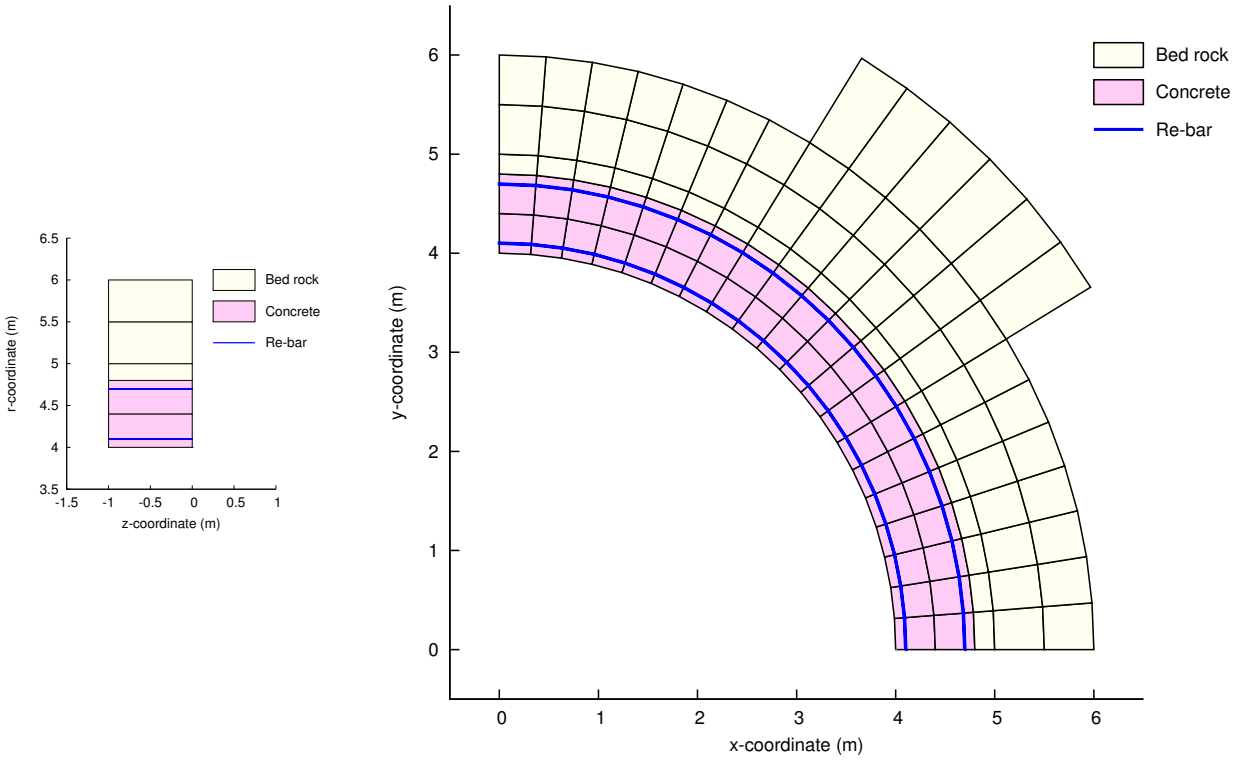
Ratio ①/②

E_g	σ_{cr}	σ_{sa}	σ_{sb}	$\sigma_{\theta g}$	σ_{rg}	u_a	u_b
1	1.000	1.001	1.000	1.000	1.000	1.001	1.001
10	1.000	1.001	1.000	1.000	1.000	1.001	1.001
100	1.000	1.001	1.000	1.000	1.000	1.001	1.001
1000	1.000	1.001	1.000	1.002	1.000	1.001	1.001
10000	1.000	1.000	1.000	1.000	1.001	1.000	1.002
100000	1.000	1.000	1.000	1.000	1.003	1.000	1.000



Axisymmetric FEM model **2D plane strain model**

Fig.1 Model for analysis of whole area (wall thickness: 800mm)



Axisymmetric FEM model **2D plane strain model**

Fig.2 Model for analysis around concrete wall
(wall thickness :800mm, double reinforcement bars)